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Anomalies and Efficient Portfolio Formation

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The Research Foundation's mission is to encourage education for investment practitioners worldwide and to fund, publish, and distribute relevant research.

Biographies

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Jay Shanken is the Dean's Distinguished Professor of Finance at Emory University's Goizueta Business School. He is an advisory editor for the *Journal of Financial Economics* and has been an associate editor for the *Journal of Finance*, *Review of Financial Studies,* and *Financial Analysts Journal*. Professor Shanken has twice received the Roger F. Murray Prize of the Institute for Quantitative Research in Finance and was a Batterymarch Fellow. He has consulted for Morgan Stanley, Barra, Treynor Arbit Associates, Acadian Asset Management, Aeltus Investment Management, and Barclays Capital. Professor Shanken's research has focused on the theory and testing of asset-pricing models and market efficiency, the predictability of stock returns, portfolio management and performance evaluation, and Bayesian econometrics. His articles have appeared in leading finance journals, including the *Journal of Finance*, *Econometrica*, *Review of Financial Studies,* and *Journal of Financial Economics*. Professor Shanken holds a B.S. in mathematics from the State University of New York at Stony Brook, an M.A. in mathematics from Cornell University, and a Ph.D. in economics from Carnegie-Mellon University (see also www.bus.emory.edu/jshanken/).

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Foreword

Financial economists have long been interested in testing the efficient market hypothesis, and as a result, they have uncovered a wide variety of apparent anomalies. For the most part, however, they have stopped short of describing how best to incorporate apparent market anomalies into efficient and practical investment portfolios. Rather, they have devised simple trading rules merely to demonstrate the presence of an anomaly. S.P. Kothari and Jay Shanken extend the market anomaly literature to its practical conclusion. Specifically, they show how to tilt portfolios away from market indexes to capture the historical alpha and residual risk associated with one or more anomalies. Moreover, they show how to discount historical parameters to reflect the possible influence of data snooping and survivorship bias. Finally, they provide explicit direction for blending prior beliefs with historical evidence by using Bayesian analysis.

Although Kothari and Shanken devote a substantial part of the monograph to documenting the existence of the value and momentum anomalies and the disappearance of the size anomaly, they are quick to point out their agnosticism with respect to the future persistence of these phenomena. Their important contribution is not reinforcement or contradiction of previously documented anomalies; rather, it is methodological. They demonstrate how best to act on evidence and prior beliefs in a manner that is true to mean–variance portfolio formation. It is up to investors to evaluate the evidence of market anomalies and weigh it against their beliefs.

This issue of the existence of market anomalies deserves particular attention. Consider the following thought experiment. Suppose the market is perfectly efficient in such a way that alphas fluctuate randomly around a mean of zero. What is the likelihood that a particular attribute, such as value, will appear to be associated with anomalous returns? If we accept the standard threshold of 95 percent, there is a 5 percent chance that a randomly selected attribute will appear to be significantly associated with a positive alpha. Now, suppose we consider another uncorrelated attribute. The chance that *both* attributes will appear to be unassociated with a positive alpha equals 95 percent squared or 90.25 percent. Therefore, the likelihood that at least one of the two attributes will appear significant equals 1.00 – 0.9025, or 9.75 percent. If we extend this logic, we find that there is a better than even chance that at least one of 14 independent attributes ($1 - 0.95^{14} = 51.23$ percent) will appear significant merely by random process—not because a true association exists between the attribute and alpha. Now, consider all the academics faced with the publish or perish reality of the tenure system, as well as all the Ph.D.

candidates in search of a dissertation topic, who diligently mine the Compustat and CRSP tapes for apparent anomalies. Should we be surprised when anomalies appear?

Kothari and Shanken are careful to reserve judgment about the durability of perceived anomalies. Instead, they provide an invaluable guide for constructing portfolios that optimally balance evidence of market anomalies with convictions about market efficiency. The Research Foundation is especially pleased to present *Anomalies and Efficient Portfolio Formation.*

Mark Kritzman, CFA
Research Director
The Research Foundation of the
Association for Investment Management and Research

Anomalies and Efficient Portfolio Formation

How an investor should combine financial investments in an overall portfolio to maximize some objective is an issue that is fundamental both to financial practice and to understanding the process that determines prices in a financial market. A key principle underlying modern portfolio theory is that bearing portfolio risk is pointless unless it is compensated for by a higher level of expected return. This principle is formalized in the concept of a mean–variance-efficient portfolio, one that has as high a level of expected return as possible for the given level of risk and incurs the minimum risk needed to achieve that expected return.

Although efficiency is an appealing concept, the appropriate composition of an efficient portfolio is far from obvious. The classic theory of risk and return, the capital asset pricing model (CAPM), provides a starting point. It implies that the value-weighted market portfolio of financial assets should be efficient. The empirical evidence accumulated since about 1980 indicates, however, that stock indexes, such as the S&P 500 Index, are not (mean–variance) efficient. This literature has uncovered various company characteristics that are significantly related to expected returns beyond what would be warranted by their contributions to the risk of the market index. Whether these "anomalies" result from limitations of the theory or the use of a stock market index in place of the true market portfolio, the practical implication is that one can construct portfolios that dominate the simple market index.

Surprisingly, not much of the work exploring the empirical limitations of the CAPM has adopted the perspective of optimal portfolio formation. Rather, the focus has been on measuring the magnitude of risk-adjusted expected returns.[1] In this monograph, we consider the implications for the efficiency of portfolios of the three most prominent CAPM anomalies—expected return effects that are negatively related to company size (market capitalization), effects that are positively related to company book-to-market ratios, and effects that are positively related to past-year momentum. For each anomaly, we estimate the amount that investors should tilt their portfolios away from the

[1]Two notable exceptions are the work of Pastor (2000), which is closely related to our analysis, and the work of Haugen and Baker (1996).

market index—toward the anomaly-based portfolio (or spread)—to exploit the gains to efficiency.[2] The same principles of modern portfolio theory can be applied, however, to other active strategies that are expected to generate positive risk-adjusted returns (e.g., an earnings-based strategy, an accruals strategy, or a trading-volume-based strategy).

The portfolio improvement obtained by tilting an index toward an active strategy depends not only on the risk-adjusted expected return of the strategy but also on residual risk (i.e., that portion of risk that depends on return variation unrelated to variation in the market index returns). This risk measure has received little attention in the academic literature, but it is important for asset allocation. We follow up on the performance of each strategy in both the first and second years after portfolio formation to get a rough indication of the relevance of portfolio rebalancing.

We also examine optimal portfolio strategies that simultaneously exploit all three anomalies and the market index. Our focus on the three most prominent anomalies should not be interpreted as suggesting that we believe these anomalies will persist in the future. Each investor will have his or her own beliefs about the likely performance of these and other strategies.

Traditional statistical tests of significance, although useful in many contexts, are not particularly well suited to investment decision making. We provide a basic introduction to Bayesian statistical methods that in recent years have achieved increasing prominence in addressing portfolio investment problems.[3] Part of the appeal of the Bayesian perspective is that it provides the analyst or investor with a rigorous framework in which to combine somewhat qualitative judgments about future returns with the statistical evidence in historical data. Such judgments, or "prior beliefs," might be based on, for example, an analyst's views concerning the ability of financial markets to efficiently process information and the speed with which this processing occurs. Related opinions about the extent to which expected returns are compensation for risk or, instead, induced by mispricing and behavioral biases are also relevant. Although the academic literature in this area sometimes focuses on very technical mathematical issues, the main ideas are fairly simple and intuitive. We hope our introduction will bring the reader close to the state of the art fairly quickly.

[2]For a practical guide to implementing an active portfolio management strategy that is grounded in modern finance, see Waring, Whitney, Pirone, and Castille (2000).

[3]See Kandel and Stambaugh (1996).

We then apply these methods in our portfolio analysis of expected return anomalies. Finally, in a conclusion to the monograph, we summarize our findings, discuss their implications, and suggest directions for future work.

Efficient Portfolios in a CAPM World

In this section, we review the fundamental principles of investment theory and their implications for efficient-portfolio formation. We discuss portfolio theory, the CAPM, and the efficient market hypothesis.[4]

Portfolio Theory. In a mean–variance setting, a risk-averse investor's utility increases with the mean and decreases with the variance of overall portfolio returns.[5] The mean is the expected return on the portfolio; the variance is the measure of the portfolio's total risk. The efficient frontier is defined as the set of portfolios with the highest expected return for each given level of portfolio return variance. Thus, modern portfolio theory implies that to maximize expected utility, an investor should choose a portfolio on the efficient frontier. In 1952, Harry Markowitz developed optimization techniques for deriving the efficient frontier of risky assets. The inputs to this derivation are estimated values of expected return, standard deviation of return, and pairwise covariances (or correlations) for the given risky securities.

An investor's portfolio selection problem is simplified by the availability of a risk-free asset. An opportunity to invest in both risky and risk-free assets implies that all efficient portfolios consist of combinations of the risk-free asset and a unique "tangency" portfolio of the risky assets. Investors who are relatively more risk averse will invest a larger fraction of their assets in the risk-free asset, whereas relatively more risk-tolerant investors will opt for a greater fraction of their investment in the tangency portfolio. All of these combinations of the tangency portfolio and the risk-free asset lie on a straight line when expected return is plotted against standard deviation of return. This line, called "the capital market line," is the efficient frontier and represents the best possible combinations of portfolio expected return and standard deviation.

[4]For a detailed treatment of the concepts in this section, see Chapters 6–9 and Chapter 12 in Bodie, Kane, and Marcus (2002) or Chapter 4 and Chapters 6–9 in Sharpe, Alexander, and Bailey (1999).

[5]More-sophisticated approaches take into account potential hedging demands for securities when the characteristics of the investment opportunity set change over time (e.g., Merton 1973 and Long 1974). Consideration of these issues is beyond the scope of this monograph.

The CAPM. The CAPM of Sharpe (1964) and Lintner (1965) builds on Markowitz's portfolio ideas and further simplifies an investor's optimal portfolio decision. The CAPM is derived with the additional critical assumption that investors have homogenous expectations, which means that all market participants have identical beliefs about securities' expected returns, standard deviations, and pairwise covariances. With homogenous expectations and the same investment horizon, all investors arrive at the same efficient frontier. Therefore, they should hold combinations of the same tangency portfolio and the risk-free asset. Because total investor demand for assets must equal the supply, the tangency portfolio in equilibrium is the value-weighted portfolio of all risky assets in the economy (i.e., the market portfolio).

The CAPM gives rise to a mathematically elegant relationship between the expected rate of return on a security and its risk measured relative to the market portfolio. Specifically, the theory implies that expected return is an increasing linear function of its covariance risk, or beta. Beta is defined as

$$\beta_i = \frac{\mathrm{cov}(R_i, R_m)}{\mathrm{var}(R_m)}, \tag{1}$$

where $\mathrm{cov}(R_i,R_m)$ is the covariance of security i's return with the return on the market portfolio and $\mathrm{var}(R_m)$ is the variance of the return on the market portfolio. Beta is identical to the (true) slope coefficient in the regression of i's returns on those of the market and thus reflects the sensitivity of security i to aggregate market movements.

The CAPM linear risk–return relationship is

$$E(R_i) = R_f + \beta_i[E(R_m) - R_f], \tag{2}$$

where $E(R_i)$ is security i's expected rate of return, R_f is the risk-free rate of return, and $[E(R_m) - R_f]$ is the market risk premium.

In addition to its importance in portfolio analysis, beta is often used to obtain discount rates for corporate valuation and real investment (i.e., capital-budgeting) decisions.

Efficient Market Hypothesis. The efficient market hypothesis states that security prices rapidly and accurately reflect all information that is available at a given point in time.[6,7] In practice, analysts and theorists focus primarily on publicly available information. Security markets are expected to

[6]For detailed reviews of the efficient market hypothesis and empirical literature on market efficiency, see Fama (1970, 1991) and MacKinlay (1997).

[7]The notion of *informational* financial market efficiency should not be confused with the earlier concept of the *mean–variance* efficiency of a portfolio.

tend toward (informational) efficiency because a large number of market participants actively compete among themselves to gather and process information and to trade on that information. Ideally, this process quickly moves the prices of securities to their "fundamental values." In a relatively efficient market, rewards to technical analysis and fundamental analysis are limited by this competition and are available only to those analysts or traders with some sort of comparative advantage. In the short run, prices may not completely adjust to new information because of various trading costs. More generally, markets may be inefficient because of behavioral biases in investor beliefs (excessive optimism or pessimism, overconfidence, etc.). Deviations from efficiency can persist if, in betting that the inefficiency will be corrected over a given horizon, arbitrageurs are exposed to substantial risk that the "mispricing" will get worse before it gets better (Shleifer and Vishny 1997).

Portfolio theory, the CAPM, and the efficient market hypothesis jointly have remarkably simple implications for investors' optimal portfolio decisions: Investors should hold a combination of the risky market portfolio and the risk-free asset, and the investment approach should be a passive buy-and-hold index strategy.[8] The picture is less clear, however, if investors believe that the CAPM does not hold and if they doubt market efficiency. We explore the attendant complexities in later sections of this monograph.

A large body of evidence suggests that security returns exhibit significant predictable deviations from the CAPM and that the capital markets are inefficient in certain respects. As discussed in detail later, these CAPM deviations, or risk-adjusted returns, are captured by a statistical parameter referred to as "Jensen's alpha."

Investors' views about these capital market issues can have important implications for the investors' optimal portfolio decisions by affecting their confidence that positive alphas observed in the past will persist in the future.

Evidence Challenging Market Efficiency

In this section, we summarize evidence on informational inefficiency in the U.S. and international capital markets. Some of the evidence suggests that information about underlying economic fundamentals can take several years to be fully reflected in stock prices. This evidence of apparent mispricing has implications for an investor's optimal portfolio decisions: Informed investors should tilt their portfolios away from the market portfolio and in a direction

[8]The proportion of assets invested in the market portfolio is a function of the investor's risk tolerance, which may change endogenously with wealth.

that exploits the inefficiency. The optimal extent of such tilting will depend on risk as well as expected reward.

Return Predictability in Short-Window Event Studies. Overwhelming evidence indicates that security prices rapidly adjust to reflect new information reaching the market.[9] Starting with Fama, Fisher, Jensen, and Roll (1969), short-window event studies have documented the market's quick response to new information. This research analyzed large samples of companies experiencing a wide range of events, such as stock splits, merger announcements, management changes, dividend announcements, and earnings releases. The evidence suggests that the market reacts within minutes to public announcements of company-specific information, such as earnings and dividends, and to macroeconomic information, such as inflation data and changes in interest rates. Rapid adjustment of prices to new information is consistent with market efficiency, but efficiency also requires that this response be, in some sense, rational or unbiased. If both conditions hold, any opportunity to benefit from the news is short-lived and investors earn only a normal rate of return thereafter.

Longer-Horizon Return Predictability. Since the early 1980s, a large body of academic and practitioner research has challenged market efficiency.[10]Mounting evidence suggests that short-term revisions in beliefs in response to new information do not always reflect unbiased forecasts of future economic conditions and that prices may take several years to incorporate the full impact of the news. Thus, long-term abnormal expected returns may be possible for an informed investor who tries to profit from the gradual price correction.

Behavioral finance models of investor behavior hypothesize systematic under- or overreaction to corporate news as a result of investors' behavioral biases or limited information-processing capabilities. These models often draw on experimental evidence from cognitive psychology and related fields. In particular, Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998, 2001), and Hong and Stein (1999) developed models to explain the apparent predictability of stock returns at various horizons.

The representativeness bias (Kahneman and Tversky 1982) causes people to overweight information patterns observed in past data that might simply be random. Because the patterns are not really descriptive of the true properties

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[9]For an excellent summary of this research, see Bodie, Kane, and Marcus (2002), Chapters 12 and 13.

[10]The discussion in this section draws on Fama (1998).

of the underlying process, they are not likely to persist. For example, investors might extrapolate a company's past history of high sales growth too far into the future and thus overreact to sales news (Lakonishok, Shleifer, and Vishny 1994; DeBondt and Thaler 1985, 1987).

Additionally, as a result of conservatism bias, investors may be slow to update their beliefs in the face of new evidence (Edwards 1968). This delay can contribute to investor underreaction to news and lead to short-term momentum in stock prices (e.g., Jegadeesh 1990 and Jegadeesh and Titman 1993). Post-earnings-announcement drift (i.e., the tendency of stock prices to drift in the direction of earnings news for 3–12 months following an earnings announcement) could also be a consequence of the conservatism bias (e.g., Ball and Brown 1968 and Litzenberger, Joy, and Jones 1971).

Stock price over- and underreaction may also be an outcome of two other human-judgment biases—investor overconfidence and biased self-attribution. Overconfident investors place too much faith in their private information about the company's prospects and thus overreact to it. In the short run, overconfidence and attribution bias (in which contradictory evidence is viewed as resulting from chance) together result in a continuing overreaction to the initial private information that induces momentum. Overconfidence about private information also causes investors to downplay the importance of publicly disseminated information. Therefore, information releases such as earnings announcements can also generate incomplete price adjustments. Subsequent earnings outcomes eventually reveal the true implications of the earlier evidence, however, resulting in predictable price reversals over long horizons.

In summary, behavioral finance theory shows how investor biases can contribute to security price over- and underreaction to news events. The existing evidence suggests that it can take up to several years for the market to correct the initial error in its response to news events.

These conclusions, however, should be viewed with some skepticism. The behavioral theories have, for the most part, been created to "fit the facts." Initially, overreaction was advanced as the main behavioral bias relevant to financial markets. Only after the strong evidence of momentum at shorter horizons became widely acknowledged were the more-sophisticated theories developed.

As just discussed, current explanations for momentum range from underreaction to short-term continuing overreaction. Thus, identifying a particular behavioral "paradigm" is difficult at this point. Moreover, work by Lewellen and Shanken (2002) demonstrates that anomalous-looking patterns in returns can also arise in a model in which fully rational investors gradually learn about certain features of the economic environment. These patterns are observed in the data with hindsight but could not have been exploited by investors in real time. Clearly, sorting out all these issues is challenging.[11]

Next, we review the evidence indicating return predictability. We caution the reader, however, that in addition to the unresolved theoretical issues, there is no consensus among academics about how to interpret the existing empirical evidence. In particular, Fama (1998) argued that much of the evidence on abnormal long-run return performance is questionable because of methodological limitations and the general effect of data mining.

Evidence of Return Predictability. Researchers have found long-horizon predictability of returns following a variety of corporate events and extreme past security price performance. The corporate events include stock splits, share repurchases, extreme earnings performance announcements, bond-rating changes, dividend initiations and omissions, seasoned equity offerings, and initial public offerings (IPOs). Evidence appears in the following studies (see Fama 1998 for a detailed discussion). Fama, Fisher, Jensen, and Roll (1969) and Ikenberry, Rankine, and Stice (1996) examined price performance following stock splits; Ibbotson (1975) and Loughran and Ritter (1995) studied post-IPO price performance; Loughran and Ritter documented negative abnormal returns after seasoned equity offerings; Asquith (1983) and Agrawal, Jaffe, and Mandelker (1992) estimated bidder companies' price performance; dividend initiations and omissions were examined by Michaely, Thaler, and Womack (1995); performance following proxy fights was studied by Ikenberry and Lakonishok (1993); Ikenberry, Lakonishok, and Vermaelen (1995) and Mitchell and Stafford (2000) examined returns following open market share repurchases; Litzenberger, Joy, and Jones (1971), Foster, Olsen, and Shevlin (1984), and Bernard and Thomas (1990) studied post-earnings-announcement returns.

The main conclusion from these studies is that the magnitude of abnormal returns is in many cases not only statistically highly significant but also economically large. From the standpoint of optimal portfolio decisions and investment strategy, however, predictable returns following corporate events provide limited opportunity to exploit the inefficiency because, typically, only a few companies experience an event each month.

Fortunately, research also shows that a small number of general company characteristics can be used to successfully predict average future returns. Moreover, a large number of securities share the characteristics that are correlated with high returns. The availability of a large pool of securities to

[11] See also Brav and Heaton (2002).

invest in reduces the loss of diversification entailed in trying to exploit the characteristic-based return predictability.

The company characteristics most highly associated with future returns are the ratio of book value to market value (BV/MV), company size, and past price performance (or momentum). As for size, Banz (1981) and, more recently, Fama and French (1993) provided evidence that small (low-market-cap) companies earn positive CAPM risk-adjusted returns. That is, small-company portfolios exhibit a positive Jensen's alpha.[12]

In relation to BV/MV, Rosenberg, Reid, and Lanstein (1985) and Fama and French (1992) showed that when "value" is based on BV/MV, value stocks significantly outperform growth stocks. The average return for the highest decile of stocks ranked according to BV/MV is almost 1 percent a month more than for the lowest decile of stocks. The Jensen's alpha of value (growth) stocks is both economically and statistically significantly positive (negative).[13]

One explanation for the superior performance of value stocks is that their high expected return reflects compensation for some sort of distress-related factor risk. An alternative interpretation is that growth stocks are overpriced "glamour" stocks that subsequently earn low returns (Lakonishok, Shleifer, and Vishny 1994; Haugen 1995).

A large literature has examined whether past price performance predicts future returns. The evidence for price *reversal* at short intervals (up to a month) and over longer horizons (three to five years) is mixed.[14] The most compelling evidence supports price *momentum* at intermediate intervals—that is, 6–12 months (Jegadeesh and Titman 1993). Only this momentum effect appears to be robust (in the post-1940 period) to the form of risk adjustment and other technical considerations we incorporate; hence, we examine the extent to which an investor can improve portfolio efficiency by tilting the optimal portfolio to exploit such price momentum.

We have identified three characteristic-based investment strategies that historically have produced positive abnormal returns. In the next section, we present mean–variance optimization techniques that can be used to exploit the abnormal-return-generating ability of these anomaly-based investment

[12] Handa, Kothari, and Wasley (1989) and Kothari, Shanken, and Sloan (1995) showed that the size effect is mitigated when portfolios' CAPM betas are estimated from annual returns.

[13] Kothari et al. showed that the BV/MV effect is attenuated considerably among the larger stocks and in industry portfolios.

[14] For price reversals at short intervals, see Jegadeesh (1990), Lehmann (1990), and Ball, Kothari, and Wasley (1995). For the longer horizon, see DeBondt and Thaler (1985, 1987), Chan (1988), Ball and Kothari (1989), Chopra, Lakonishok, and Ritter (1992), Lakonishok, Shleifer, and Vishny (1994), and Ball, Kothari, and Shanken (1995).

strategies. The optimization analysis is intended to serve only as a guiding tool for investment managers, however, by highlighting the potential impact of tilt strategies on portfolio risk and return. In general, managers will also be guided by their own research, their beliefs about the likelihood that historically successful strategies will continue to perform well in the future, the market conditions prevailing at the time of their investment decisions, and other factors, such as transaction costs, international diversification, and taxes.

Implications for Optimal Portfolio Decisions. Evidence of market inefficiency often translates into investment strategies that have significant nonzero CAPM alphas. The intuitive implication for optimal portfolio formation is to tilt the investment portfolio away from the passive market portfolio and toward the positive-alpha investment strategy. The extent of the tilt should increase in the magnitude of the expected abnormal return from the strategy. Such a tilt, however, typically exposes the investor to residual risk that reflects return variation unrelated to the market index returns. So, the greater the residual risk, the less the recommended tilt. The optimal portfolio decision that accounts for the magnitude of potential abnormal return as well as the residual risk incurred was formally derived in a classic paper by Treynor and Black (1973) and is discussed in the next section.

Optimal Portfolio Tilts

We present evidence in this section on the historical performance in the U.S. market of small-cap versus large-cap stocks, value versus growth stocks, and positive-momentum versus negative-momentum stocks for four decades. Using this evidence to form expectations about future performance, we then discuss the extent to which tilting an investor's portfolio in favor of various anomaly-based strategies improves an investor's risk–return trade-off. We estimate optimal portfolio decisions under a variety of assumptions about the investor's beliefs concerning the efficiency of a market index and the profitability of investing in size, value, or momentum strategies. For example, the investor might believe that the historical alpha of these stocks overstates their forward-looking alpha because of a combination of factors, including data snooping, survival biases, and chance.

Although an analysis of the sort presented here can provide useful guidance about optimal portfolio decisions, quantitative optimization techniques should not be viewed as black boxes that produce uniquely correct answers. Far too many assumptions go into any such optimization, and judgment will always play an important role in the portfolio decision. Modern portfolio techniques can be an important tool, however, for enhancing that judgment. From this perspective, we believe it is important to first explore the optimal-tilt problems in detail for each of the strategies considered here. Going through this process will give the reader a good feel for the basic historical risk–return characteristics of these strategies in conjunction with a simple index strategy. In later discussions, we provide some additional results for optimal portfolios based on simultaneous optimization across several strategies. This analysis will take into account the correlations between the various returns in addition to their individual risk–return attributes.

Data. We constructed a comprehensive database of New York Stock Exchange (NYSE), American Stock Exchange (Amex), and Nasdaq equity securities for our analysis. We included all company-year observations with valid data available from the Center for Research in Security Prices (CRSP) and Compustat tapes for the 1963–99 period. We measured buy-and-hold (compounded) annual returns from July of year t to June of year $t + 1$, starting in July 1963 (for a total of 36 years). For each year, we included all companies with Compustat data available for calculating BV/MV and with CRSP data available for calculating market cap and previous-one-year return (to assign stocks to momentum portfolios). Because book-value data on Compustat are not as frequently available as return data on CRSP, only 75,272 company-year observations are in the BV/MV analysis.

We required that included securities have the size, BV/MV, and momentum information prior to calculating their annual return starting on 1 July. Specifically, we measured market cap at the end of June of year t (for example, size was measured at the end of June 1963), and returns were computed for the period from July 1963 to June 1964. Book value was measured at the end of the previous fiscal year (typically, December of the previous year; i.e., December 1962 for returns computed for July 1963 to June 1964). The December–July gap ensured that the book-value number was publicly available at the time of portfolio formation. Following Fama–French (1993), we calculated book value as the Compustat book value of stockholders' equity plus balance-sheet-deferred taxes and investment tax credit (if available) plus postretirement-benefit liability (if available) minus the book value of preferred stock. We calculated the book value of preferred stock as, depending on availability, the redemption, liquidation, or par value (in that order). BV/MV was calculated as the book value of equity for the fiscal year ending in calendar year $t - 1$ divided by the market value of equity obtained at the end of June of year t.

For studying momentum, we ranked stocks on the basis of their performance over a one-year period ending on 31 May of each calendar year and implemented the investment strategy one month later starting on 1 July.

Skipping a month avoided well-known bid–ask effects that bias momentum performance downward.

We analyzed the performance of value-weighted quintiles each year. We constructed these portfolios at the end of June of year *t* on the basis of size, BV/MV, and momentum. The portfolios based on BV/MV do not include companies with negative or zero BV/MV values. The portfolios based on momentum do not include companies that lacked return data for the 12 months preceding portfolio formation.

Some securities did not remain active for the 12-month period beginning on 1 July. Companies were delisted as a result of mergers, acquisitions, financial distress, or violation of exchange listing requirements. We included the delisting return, when available, for delisted securities as reported on the CRSP tapes. This step prevented survival bias from exaggerating an investment strategy's performance.

The empirical analysis gives consideration to the practical feasibility of mutual funds implementing the optimal portfolio recommendations in this monograph. Toward this end, therefore, we excluded from our analysis stocks with small market caps and low prices, which are impractical for investment purposes. Investments of an economically meaningful magnitude at current market prices can be difficult in small-cap stocks, which lack liquidity, and low-priced stocks, which are typically associated with high transaction costs. Therefore, we report the results of forming optimal portfolios by restricting the universe of stocks analyzed to those stocks with market capitalizations in excess of the smallest decile of stocks listed on the NYSE and prices greater than $2.

Descriptive Statistics. Table 1 reports descriptive statistics for the sample of equity securities that we assembled for optimal portfolio analysis.

Table 1. Descriptive Statistics for Entire Sample, July 1963–June 1999

Variable	Mean	Median	Standard Deviation	*N* (average per year)
Return for year *t* (%)	14.3	9.3	42.0	2,802
Market value ($ millions)	723.5	143.5	2,893.70	2,802
Price ($)	27.59	21.43	122.87	2,802
BV/MV	1.04	0.66	4.79	2,090
Return for year *t* – 1 (%)	22.8	13.9	51.6	2,560

Note: The descriptive statistics are the time-series averages of the cross-sectional statistics (number of observations, mean, median, and standard deviation) obtained every year. For momentum portfolios using return for year *t* – 1, the total number of company-years is 92,182.

The total number of company-year observations for 1963–1999 is 100,904, with an average of about 2,800 companies a year. If we had not excluded stocks priced lower than $2 or stocks in the lowest decile of the market cap of NYSE stocks, the number of securities each year would have been approximately 4,800. The cross-sectional standard deviation is three times the annual buy-and-hold return. Also, because of some spectacular winners, the median annual return is considerably lower than the mean.

The average return for year $t - 1$ (the year prior to investment), reported in the last row of Table 1, is much higher than the average return for year t. We attribute this difference to the exclusion of low-priced and small-cap stocks; stocks that were experiencing negative returns declined in price and market value by the end of year $t - 1$. The stocks retained for investment at the beginning of year t typically performed relatively well in the prior year, which naturally boosted the average return for year $t - 1$ reported in Table 1. All of our portfolio analysis is forward looking and, therefore, not subject to survivor bias.

The average market cap of the sample securities is considerably higher than the median stock's market value.[15] The mean BV/MV of 1.04 is a result of two factors. The first is the exclusion of the small-cap and low-priced stocks, many of which also had low book values because of asset write-offs, restructuring charges, and so on. Second, although BV/MVs in the 1990s were at the low end of the distribution of this measure, the ratios in the 1970s were quite high, which raised the average for our sample. The median over the full period is less than 1.0, however, well below the mean.

Definitions and Notation. To assess the performance of each strategy (small cap versus large cap, value versus growth, and momentum), we formed quintile portfolios based on 1 July data of each year by ranking all available stocks according to their market cap, BV/MV, and previous-one-year performance. We estimated each portfolio's risk-adjusted performance for the following year. We then measured the performance of a portfolio formed by tilting the value-weighted market portfolio toward the quintile portfolios, with the weight of a quintile portfolio ranging from 0 to 100 percent and that of the value-weighted market portfolio declining from 100 percent to 0. That is, the value-weighted market portfolio was gradually tilted all the way toward a quintile portfolio. The optimal tilt was achieved when the Sharpe ratio (excess return/standard deviation of excess return) of the tilt portfolio attained the maximum.

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[15]The market-cap numbers were not adjusted for inflation through time, so both real and nominal effects cause variation in market values across years.

We used a CAPM regression to estimate a portfolio's risk-adjusted performance. The estimated intercept from a regression of portfolio excess returns on the excess value-weighted market return is considered the abnormal performance, or Jensen's alpha, of the portfolio. To estimate the CAPM regression, we used the time series of annual postranking quintile portfolio returns for July 1963 to July 1999. The identity of the stocks in each quintile portfolio changed annually because all available stocks were reranked each 1 July on the basis of market capitalization, BV/MV, or previous-one-year performance. The CAPM regression was as follows:

$$R_{qt} - R_{ft} = \alpha_q + \beta_q(R_{mt} - R_{ft}) + \varepsilon_{qt}, \tag{3}$$

where

$R_{qt} - R_{ft}$ = buy-and-hold value-weighted excess return on quintile portfolio q for year t, defined as the quintile portfolio return minus the annual risk-free rate

$R_{mt} - R_{ft}$ = excess return of the CRSP value-weighted market portfolio

α_q = abnormal return (or Jensen's alpha) for portfolio q over the entire estimation period

β_q = CAPM beta risk of portfolio q over the entire estimation period

ε_{qt} = residual risk

Table 2, **Table 3**, and **Table 4** report performance for allocations tilted toward, respectively, size, BV/MV, and momentum portfolios. (The terminology used in these tables is given in **Exhibit 1**.) For example, for the size portfolios in Table 2, we report in Panel A the performance of a portfolio consisting of *X* percent of the smallest market-cap size quintile (Q1) or largest market-cap size quintile (Q5) and (100 – *X*) percent of the CRSP value-weighted portfolio. The *X* amount of the size quintile in the portfolio varies from 0 (i.e., all the investment in the market portfolio and no tilt toward a size quintile) to 100 percent (i.e., all the investment in a size quintile).

The last section of Panel A reports the results for a strategy of tilting toward the spread between the largest quintile (Q5) and the smallest quintile (Q1). Although this information is probably of less practical relevance than the other Panel A results, with the proliferation of exchange-traded funds tied to a variety of indexes, implementing such spread strategies may eventually become realistic. For an introduction to the notation and concepts used for the spread findings, we will describe the Q5–Q1 spread for the size quintiles.[16]

[16]The conventional size-based strategy of emphasizing small companies would correspond to a negative position in this spread, but the risk-adjusted performance of this small – large strategy is slightly negative for our sample.

Exhibit 1. Terminology

rt	Return to anomaly-based active quintile or spread between extreme quintiles.
exrt	Time-series average of annual excess returns on a tilt portfolio (over 36 years). Excess return is the raw portfolio return minus the 1-year risk-free rate.
0%, 20%, . . .	Percentage of *rt* in each tilt portfolio return; 0 percent corresponds to the value-weighted market (CRSP) return, and 100 percent corresponds to *rt*.
$\sigma(exrt)$	Standard deviation of excess return.
α	Intercept from a CAPM regression of excess active portfolio return on the excess market return; the alpha of each tilt portfolio is a fraction of the alpha of the active portfolio (e.g., α for a 10 percent tilt is $0.10 \times \alpha$ of *rt*).
Sharpe	Sharpe ratio; that is, $exrt/\sigma(exrt)$.
c	Measure of an investor's lack of confidence in historical performance; we used $c = 0.5$, which means that prospective asset allocations are based on 50 percent of the historical alpha estimates.
c_Sharpe	Sharpe ratio based on reduced alpha; that is, $(exrt - c \times \alpha)/\sigma(exrt)$.
M^2	*M*-square measure of performance; that is, *Sharpe* $\times\ \sigma$ (Market excess return) = Excess return on a combination of the active portfolio and the riskless asset that has the same standard deviation as the market portfolio.
*c_*M^2	*M*-square measure based on reduced alpha; that is, *c_Sharpe* $\times\ \sigma$ (Market excess return).
β	Slope coefficient from a CAPM regression of excess active portfolio return on the excess market return.
$\sigma(\varepsilon)$	Standard deviation of residuals from the CAPM regression for an active portfolio.
Shp_mkt	Sharpe ratio of the value-weighted market portfolio.
Info	Information ratio, $\alpha/\sigma(\varepsilon)$.
Shp_opt	Sharpe ratio of the optimal portfolio with unrestricted short selling and confidence level *c*; that is, $\sqrt{Shp\_mkt^2 + \{(1-c)[\alpha/\sigma(\varepsilon)]\}^2}$.

Technically, the tilt "asset" in the context of spread results should be viewed as a position consisting of $1 in U.S. Treasury bills and $1 on each side of the large-cap minus small-cap spread. In other words, the investor in this asset is implicitly assumed to receive interest on the proceeds from the short sale of the small-company quintile. This combined position has a net investment of $1, unlike the spread itself, which is a zero-investment portfolio with an undefined rate of return. Because the focus is on excess returns, the return on the $1 investment in T-bills is netted out and the performance measures are determined completely by the spread between large-cap quintile and small-cap quintile stock returns. The return calculations ignore the impact of margin requirements that might be associated with either long or short positions. If the spread portfolio generates a positive alpha, then an investor

Table 2. Performance of Portfolios Tilted toward Size Quintiles, July 1963–June 1999 Data

A. Tilt portfolios' performance

Variable	0%	20%	40%	50%	60%	80%	100%
Quintile 1							
exrt	7.4%	7.6%	7.8%	8.0%	8.1%	8.3%	8.6%
σ(*exrt*)	17.7	18.8	20.4	21.3	22.3	24.6	27.0
α	0.0	–0.1	–0.2	–0.3	–0.3	–0.4	–0.5
Sharpe	41.5	40.4	38.5	37.3	36.2	33.9	31.7
c_Sharpe	41.5	40.7	39.0	37.9	36.9	34.7	32.7
M^2	7.4	7.1	6.8	6.6	6.4	6.0	5.6
c_M^2	7.4	7.2	6.9	6.7	6.5	6.1	5.8
Quintile 5							
exrt	7.4%	7.3%	7.3%	7.3%	7.3%	7.3%	7.3%
σ(*exrt*)	17.7	17.5	17.4	17.3	17.2	17.1	17.0
α	0.0	0.1	0.1	0.1	0.2	0.2	0.3
Sharpe	41.5	41.8	42.1	42.2	42.3	42.5	42.7
c_Sharpe	41.5	41.7	41.8	41.8	41.9	41.9	41.9
M^2	7.4	7.4	7.5	7.5	7.5	7.5	7.6
c_M^2	7.4	7.4	7.4	7.4	7.4	7.4	7.4
Quintile 5 – Quintile 1							
exrt	7.4%	5.6%	3.9%	3.0%	2.2%	0.4%	–1.3%
σ(*exrt*)	17.7	13.6	11.2	11.0	11.6	14.5	18.8
α	0.0	0.2	0.3	0.4	0.5	0.6	0.8
Sharpe	41.5	41.2	34.6	27.3	18.6	2.9	–7.0
c_Sharpe	41.5	40.6	33.2	25.6	16.6	0.7	–9.1
M^2	7.4	7.3	6.1	4.8	3.3	0.5	–1.2
c_M^2	7.4	7.2	5.9	4.5	2.9	0.1	–1.6

B. 100% tilt and optimal portfolio

Variable	Q1	Q5	Q5 – Q1
α (%)	–0.5	0.3	0.8
	(2.9)	(0.4)	(3.3)
β	1.24	0.95	–0.29
	(0.15)	(0.02)	(0.18)
σ(ε) (%)	16.1	2.3	18.3
Shp_mkt (%)	41.5	41.5	41.5
	(16.7)	(16.7)	(16.7)
Info (%)	–3.2	11.5	4.2
	(18.1)	(18.1)	(18.1)
Shp_opt (%)	41.6	41.9	41.6
	(16.7)	(16.7)	(16.7)

Notes: In Panel A, $c = 0.5$; in Panel B, *Shp_opt* also is given for $c = 0.5$. In Panel B, the standard errors are given in parentheses below each estimate. For alpha and beta, the standard errors come from the regression. For Sharpe ratios and information ratios, approximate standard errors are given.

Table 3. Performance of Portfolios Tilted toward BV/MV Quintiles, July 1963–June 1999 Data

A. Tilt portfolios' performance

Variable	0%	20%	40%	50%	60%	80%	100%
Quintile 1							
exrt	7.4%	7.5%	7.7%	7.8%	7.9%	8.0%	8.2%
σ(*exrt*)	17.7	18.5	19.6	20.2	20.8	22.1	23.5
α	0.0	–0.1	–0.3	–0.4	–0.4	–0.6	–0.7
Sharpe	41.5	40.5	39.3	38.5	37.8	36.3	34.8
c_Sharpe	41.5	40.9	40.0	39.4	38.8	37.6	36.3
M^2	7.4	7.2	6.9	6.8	6.7	6.4	6.2
c_M^2	7.4	7.2	7.1	7.0	6.9	6.6	6.4
Quintile 5							
exrt	7.4%	8.0%	8.7%	9.1%	9.4%	10.1%	10.8%
σ(*exrt*)	17.7	17.2	16.8	16.7	16.7	16.7	16.9
α	0.0	0.9	1.9	2.3	2.8	3.7	4.7
Sharpe	41.5	46.8	51.8	54.2	56.5	60.6	64.0
c_Sharpe	41.5	44.1	46.3	47.2	48.1	49.4	50.1
M^2	7.4	8.3	9.2	9.6	10.0	10.7	11.3
c_M^2	7.4	7.8	8.2	8.4	8.5	8.7	8.9
Quintile 5 – Quintile 1							
exrt	7.4%	6.4%	5.5%	5.0%	4.5%	3.6%	2.6%
σ(*exrt*)	17.7	13.2	10.3	9.8	10.2	13.1	17.6
α	0.0	1.1	2.2	2.7	3.2	4.3	5.4
Sharpe	41.5	48.4	53.1	50.7	44.1	27.1	14.9
c_Sharpe	41.5	44.3	42.6	37.0	28.3	10.7	–0.5
M^2	7.4	8.6	9.4	9.0	7.8	4.8	2.6
c_M^2	7.4	7.8	7.5	6.6	5.0	1.9	–0.1

B. 100% tilt and optimal portfolio

Variable	Q1	Q5	Q5 – Q1
α (%)	–0.7	4.7	5.4
	(1.8)	(1.5)	(3.0)
β	1.21	0.83	–0.38
	(0.09)	(0.08)	(0.16)
σ(ε) (%)	9.9	8.4	16.5
Shp_mkt (%)	41.5	41.5	41.5
	(16.7)	(16.7)	(16.7)
Info (%)	–7.1	55.9	32.7
	(18.1)	(18.1)	(18.1)
Shp_opt (%)	41.7	50.1	44.6
	(16.7)	(16.7)	(16.7)

Note: See Table 2 notes.

Table 4. Performance of Portfolios Tilted toward Momentum Quintiles, July 1963–June 1999 Data

A. Tilt portfolios' performance

Variable	0%	20%	40%	50%	60%	80%	100%
Quintile 1							
exrt	7.4%	6.7%	6.1%	5.8%	5.5%	4.9%	4.3%
σ(*exrt*)	17.7	18.2	18.9	19.3	19.8	20.9	22.1
α	0.0	–0.8	–1.5	–1.9	–2.3	–3.1	–3.8
Sharpe	41.5	37.1	32.4	30.1	27.9	23.5	19.5
c_Sharpe	41.5	39.2	36.5	35.1	33.7	30.9	28.2
M^2	7.4	6.6	5.7	5.3	4.9	4.2	3.4
c_M^2	7.4	6.9	6.5	6.2	6.0	5.5	5.0
Quintile 5							
exrt	7.4%	8.3%	9.2%	9.6%	10.1%	11.0%	11.9%
σ(*exrt*)	17.7	18.0	18.5	18.9	19.3	20.3	21.4
α	0.0	0.8	1.7	2.1	2.5	3.3	4.1
Sharpe	41.5	45.8	49.4	50.8	52.1	54.0	55.4
c_Sharpe	41.5	43.5	44.9	45.3	45.6	45.8	45.7
M^2	7.4	8.1	8.7	9.0	9.2	9.6	9.8
c_M^2	7.4	7.7	7.9	8.0	8.1	8.1	8.1
Quintile 5 – Quintile 1							
exrt	7.4%	7.4%	7.4%	7.5%	7.5%	7.5%	7.6%
σ(*exrt*)	17.7	14.4	12.4	12.0	12.3	14.2	17.4
α	0.0	1.6	3.2	3.2	4.8	6.4	8.0
Sharpe	41.5	51.4	60.2	61.9	60.9	53.0	43.4
c_Sharpe	41.5	45.8	47.3	45.3	41.4	30.5	20.5
M^2	7.4	9.1	10.7	11.0	10.8	9.4	7.7
c_M^2	7.4	8.1	8.4	8.0	7.3	5.4	3.6

B. 100% tilt and optimal portfolio

Variable	Q1	Q5	Q5 – Q1
α (%)	–3.8	4.1	8.0
	(1.9)	(2.0)	(3.2)
β	1.11	1.05	–0.06
	(0.10)	(0.10)	(0.17)
σ(ε) (%)	10.3	10.8	17.6
Shp_mkt (%)	41.5	41.5	41.5
	(16.7)	(16.7)	(16.7)
Info (%)	–37.2	38.3	45.3
	(18.1)	(18.1)	(18.1)
Shp_opt (%)	45.5	45.7	47.3
	(16.7)	(16.7)	(16.7)

Note: See Table 2 notes.

can improve performance by tilting toward the spread. In this case, an X percent "tilt toward the spread" entails a $\$(100 - X)$ investment in the CRSP value-weighted (market) portfolio and $\$X$ in the spread asset.

Results. Tables 2, 3, and 4 report a variety of statistics in the A panels for each tilt portfolio. As we report the table results, readers may also consult the figures containing plots of the behavior of three portfolio performance metrics —excess returns (*exrt*), M^2, and c_M^2, as defined in Exhibit 1—for portfolio allocations tilted toward Q1 and Q5 portfolios and the Q5–Q1 spread. The graphical presentation of the information helps in visualizing the costs and benefits of tilting toward various investment strategies. The figures also help convey the performance metric's sensitivity to small deviations from the optimal portfolio. If the sensitivity is low, the potential loss in performance for moderate deviations from the optimum is not great. Given the inherent limitations of any analysis of this sort, our confidence in the relevance of the results would be substantially reduced if too much sensitivity were observed.

■ *Tilting toward size quintiles.* The top section of Table 2's Panel A considers tilts toward small-cap stocks. The first column of Table 2 shows an average annual excess return of 7.4 percent for the CRSP value-weighted portfolio (0 percent tilt) for the sample period, which can be compared with the average 8.6 percent return for the 100 percent investment in Q1. Panel A of **Figure 1** illustrates the difference in excess returns.

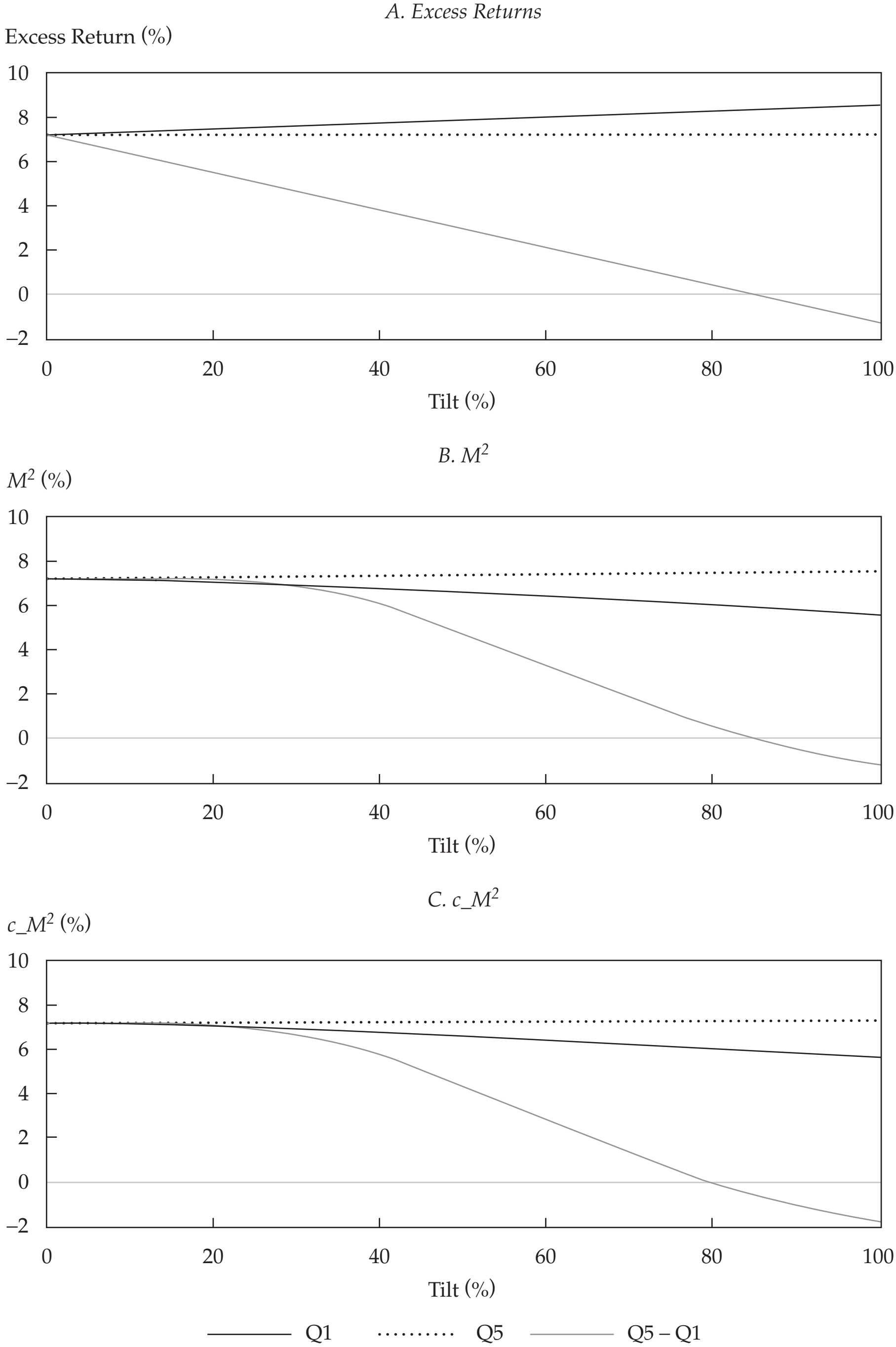
Note that the size effect (Banz) is not observable in this sample: Increasing investment in Q1 generates decreasing alpha—down to the –0.5 percent for the 100 percent investment in Q1, for which Panel B reports a standard error of 2.9 percent and a beta of 1.24 (standard error of 0.15). The poor performance of small-cap stocks in the 1980s and our decision to exclude extremely low priced and low-market-cap stocks resulted in the negative alpha for the Q1 portfolio. Without our data screen, the Q1 portfolio alpha would have been 3.3 percent.

As Panel A of Table 2 demonstrates, tilting toward small-cap stocks lowers the Sharpe ratio from the market portfolio's 41.5 percent. Because tilting toward the Q1 portfolio increases volatility faster than the increase in average returns for the 1963–99 period, the value-weighted market portfolio has the optimal Sharpe ratio.

An equivalent measure of portfolio performance that some analysts prefer to report is M^2, which (as defined in Exhibit 1) is the excess return on a portfolio after an adjustment to make its volatility equal to that of the market.[17]

[17] M^2 is named after Franco and Leah Modigliani. They introduced the measure in their 1997 *Journal of Portfolio Management* article.

Figure 1. Size Portfolio Tilts, July 1963–June 1999 Data



Formally, M^2 is the excess return to a hypothetical portfolio, $p*$, that takes positions in T-bills as well as in the given portfolio in such a way that the return volatility of $p*$ is the same as the volatility of the value-weighted market portfolio. For example, if a size-quintile-tilted portfolio's volatility (standard deviation of return) exceeds that of the market, then $p*$ includes a long position in T-bills to lower the risk.[18] The excess return on the resulting portfolio is referred to as M^2. The market portfolio's M^2 is simply its excess return, which serves as the benchmark that the investor hopes to beat by exploiting active positions in, for instance, anomaly-based portfolios.

M^2 is a positive linear transformation of the Sharpe ratio; hence, the two performance measures provide identical rankings of portfolios. But because its unit of measurement is percentage excess return, M^2 may be a more intuitive measure than the Sharpe ratio. Table 2 and Panel B of Figure 1 show that tilting toward the Q1 portfolio lowers the M^2 in relation to the market portfolio. In the extreme, the M^2 of the Q1 portfolio is 5.6 percent, a decrease of 1.8 percentage points from the market portfolio's 7.4 percent excess return.

The two additional performance measures, *c_Sharpe* and c_M^2, capture an investor's skepticism about the historical performance of an investment strategy through a variable, c, that modifies the historical alpha. The modification entails placing weights of c on 0 and $1 - c$ on the historical estimate. An investor might believe that historical performance is exaggerated because of data snooping, survivor biases, or luck or because the investment opportunity will be arbitraged away in the future as a result of public knowledge of the opportunity. In addition, a priori theoretical considerations may incline the investor to discount the historical evidence somewhat. We elaborate on this point later. The lower an investor's confidence that the past performance of an investment strategy will persist, the larger the value of c. We generally report results under the assumption that only half of the historical alphas of a portfolio can be expected in the future (i.e., $c = 0.5$).

The results shown in Table 2 for a strategy of tilting toward Quintile 1 with $c = 0.5$ indicate, not surprisingly, that tilting remains unattractive. The *c_Sharpe* measure of the Q1 portfolio has risen somewhat to 32.7 percent (compared with 31.7 percent without the c adjustment) but remains well below the 41.5 percent for the value-weighted portfolio. The slight improvement coming from the c adjustment results from the estimated small-company alpha being negative.

[18] In the opposite situation, leverage (shorting the riskless asset) would be used to raise the volatility of $p*$ to that of the market index. In that case, performance would be overstated insofar as the borrowing rate exceeded the T-bill rate used in the computation.

In addition to the performance of a series of portfolios tilted toward Q1, we report in the last row of Panel B in Table 2 performance for the optimal tilt (with $c = 0.5$) in the absence of short-selling constraints. In the terminology we are using, the Sharpe ratio of the optimal portfolio is (Treynor and Black)

$$Shp\_opt = [Shp\_mkt^2 + Info^2]^{1/2}. \tag{4}$$

The information ratio in this equation is $\alpha/\sigma(\varepsilon)$, where α is the Jensen's alpha of the portfolio strategy that will be added to the simple index position (the reduced alpha for the Q1 portfolio with $c = 0.5$) and $\sigma(\varepsilon)$ is the standard deviation of the residuals from the CAPM regression used to estimate the α (i.e., the standard deviation estimate for ε_{qt} in Equation 3).[19] The optimal amount of tilting increases with the magnitude of the Jensen's alpha and decreases with the residual uncertainty. This pattern is logical because the investor must bear residual risk by tilting away from a simple diversified position in the market index and alpha is the reward for doing so.

As the last row in Panel B in Table 2 reports, the optimal portfolio's Sharpe ratio for small-cap tilts is 41.6 percent. Because the value-weighted market portfolio's Sharpe ratio is 41.5 percent and because tilting toward the Q1 size portfolio reduces the Sharpe ratio, an investor must short the Q1 portfolio to reach optimality. This shorting improves the Sharpe ratio only marginally, however, so the optimal strategy would essentially be simply to invest in the market portfolio.

Consider now a strategy of tilting toward Q5, as shown in the middle sections of Table 2's Panel A and Panel B. These results suggest that for the 1963–99 period, investors would have gained only slightly by investing in large-cap stocks. Even though tilting the value-weighted portfolio toward Q5 by about 90 percent maximizes the Sharpe ratio for $c = 0.5$, the M^2 of the resulting portfolio is approximately the same as that for the value-weighted portfolio, 7.4 percent (see also the flat line in Panel B of Figure 1).

Finally, as shown in the spread sections of Table 2, we find that the Q5–Q1 spread portfolio provides a small alpha (0.8 percent, with a standard error of 3.3 percent) that is not statistically significant. Not surprisingly, a tilt toward this spread portfolio does not at all improve the Sharpe ratio or the M^2 over those of the value-weighted portfolio. In fact, the graphs in Panels B and C of Figure 1 show that the M^2 measures drop dramatically for tilts of more than 25 percent or so. Thus, the optimal course is, again, to invest almost the entire portfolio in the value-weighted index.

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[19]The optimal portfolio's composition is determined from the following formula: Optimal allocation to the active portfolio = $X/[1 + (1 - \beta)X]$, where $X = [Info/Shp\_mkt] \times [\sigma(\text{Market return})/\sigma(\varepsilon)]$.

■ *Tilting toward BV/MV quintiles.* Table 3 and **Figure 2** report the results of tilting portfolios toward extreme BV/MV quintiles. Table 3 shows that investing in the highest BV/MV (i.e., value, Q5) stocks yielded a highly significant Jensen's alpha of 4.7 percent a year (standard error of 1.5 percent) for the period studied. For Quintile 5, as the CRSP portfolio is tilted toward the 100 percent Q5 stocks, the Sharpe ratio increases from 41.5 percent to 64.0 percent. The corresponding M^2 performance measure increases from 7.4 percent to 11.3 percent. The *c_Sharpe* and *c_*M^2 measures, computed with alphas cut in half, also rise but not as spectacularly.

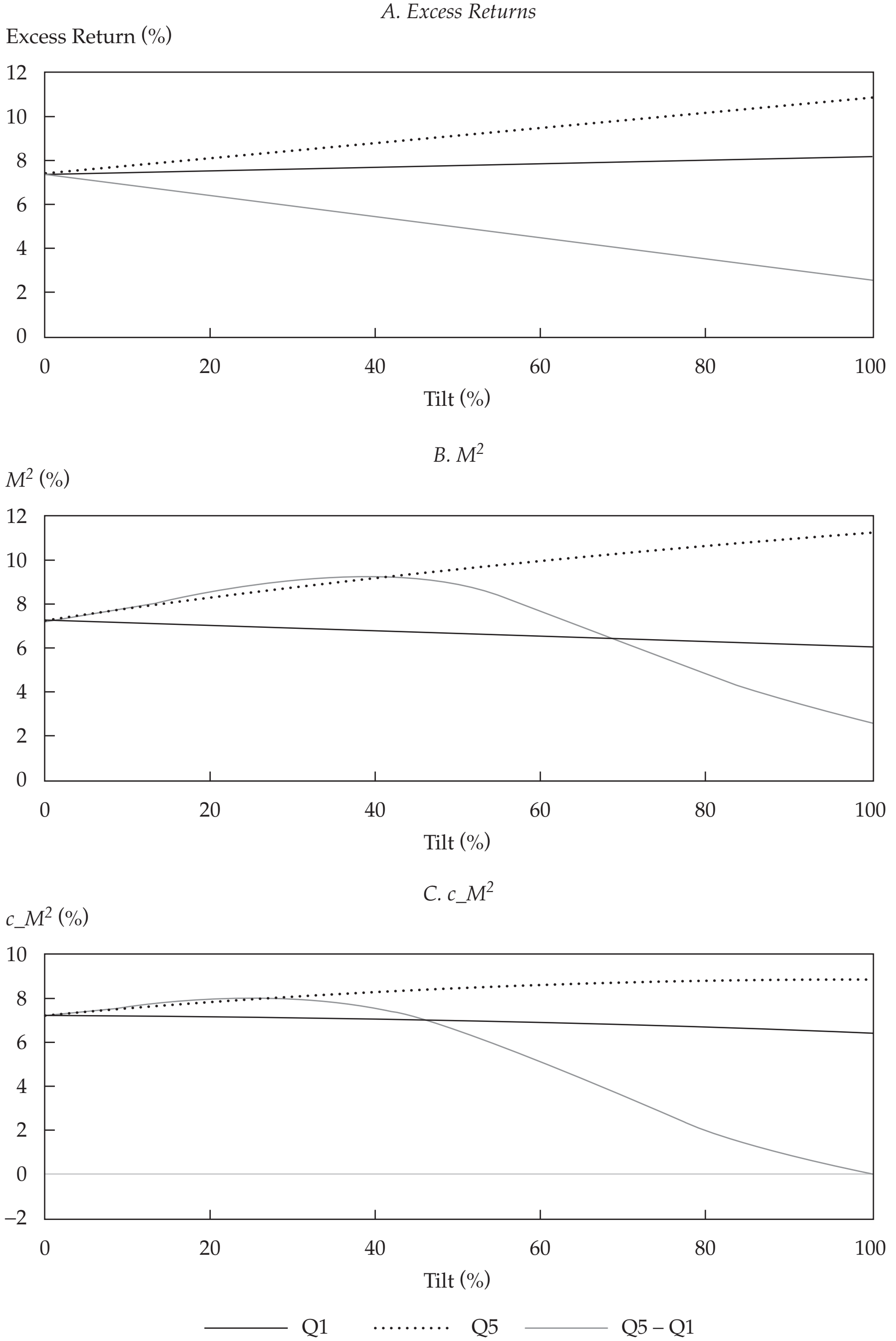
Interestingly, the optimal portfolio is fully invested in the highest-BV/MV stocks even after cutting the estimated alpha in half (*Shp_opt*). One interpretation of the superior performance of the high-BV/MV stocks is that these were distressed stocks that went on to exhibit superior performance *ex post* in the 1963–99 period. It is also possible that the high average returns reflect *ex ante* compensation for such risk. Alternatively, these stocks may have been ones that were initially undervalued and subsequently performed better than investors expected. Whatever the explanation, keep in mind that we screened out the lowest-priced and smallest-cap stocks, which enhances the practicality of investing in these quintiles in the period.

Table 3 also shows that growth stocks (i.e., low-BV/MV stocks) slightly underperformed the market, although the value-weighted alpha of –0.7 percent is statistically indistinguishable from zero. As with small companies, tilting toward growth stocks lowers the Sharpe ratio and M^2 measure (see also Figure 2, Panel B). The lack of statistical significance, however, leaves us less confident about the potential benefits from shorting the growth stocks based on this historical performance.

The Q5–Q1 spread results for BV/MV in Table 3 are notable in several respects, and we found similar results whether we imposed the small/low price filter or not. First, the optimal portfolio is invested nearly 40 percent in the spread, with the percentage dropping to about 30 when alpha was cut in half. The spread provided a large alpha (5.4 percent), and the spread beta is negative; the high-BV/MV quintile has a significantly lower beta than the low-BV/MV quintile. The fact that the average excess return declines as the spread weight is increased, despite the 5.4 percent alpha at the 100 percent level, might seem odd, but the pattern is mechanically driven by the fact that the spread return is lower than the market return. The benefit of exploiting the small (negative) alpha for growth, by shorting Q1, is dwarfed by the impact of the relatively high residual risk for growth.

Interestingly, Table 3 reports that the information ratio for the spread (Panel B) is much lower than it is for the high-BV/MV quintile, 32.7 percent

Figure 2. BV/MV Portfolio Tilts, July 1963–June 1999 Data



versus 55.9, even though shorting the low-BV/MV (negative-alpha) stocks increases the alpha a bit. The spread information ratio is lower than for Quintile 5 because the spread is exposed to much more residual risk—16.5 percent versus 8.4 percent for Q5. The residual risk of the spread would be dampened if the residual returns for value and growth were positively correlated (they would be partially hedged in the spread), but in fact, the correlation is negative. As a result, one would be much better off investing in a value strategy that emphasizes high-BV/MV stocks than trying to exploit the spread. Based on the full alpha (Panel A), the optimal M^2 values are 11.3 percent and 9.4 percent for, respectively, Q5 (100 percent in Q5) and Q5 – Q1 (40 percent in the spread). In fact, Panel B of Figure 2 clearly shows that investing 50 percent or more of the portfolio in Q5 dominates the optimal spread position.

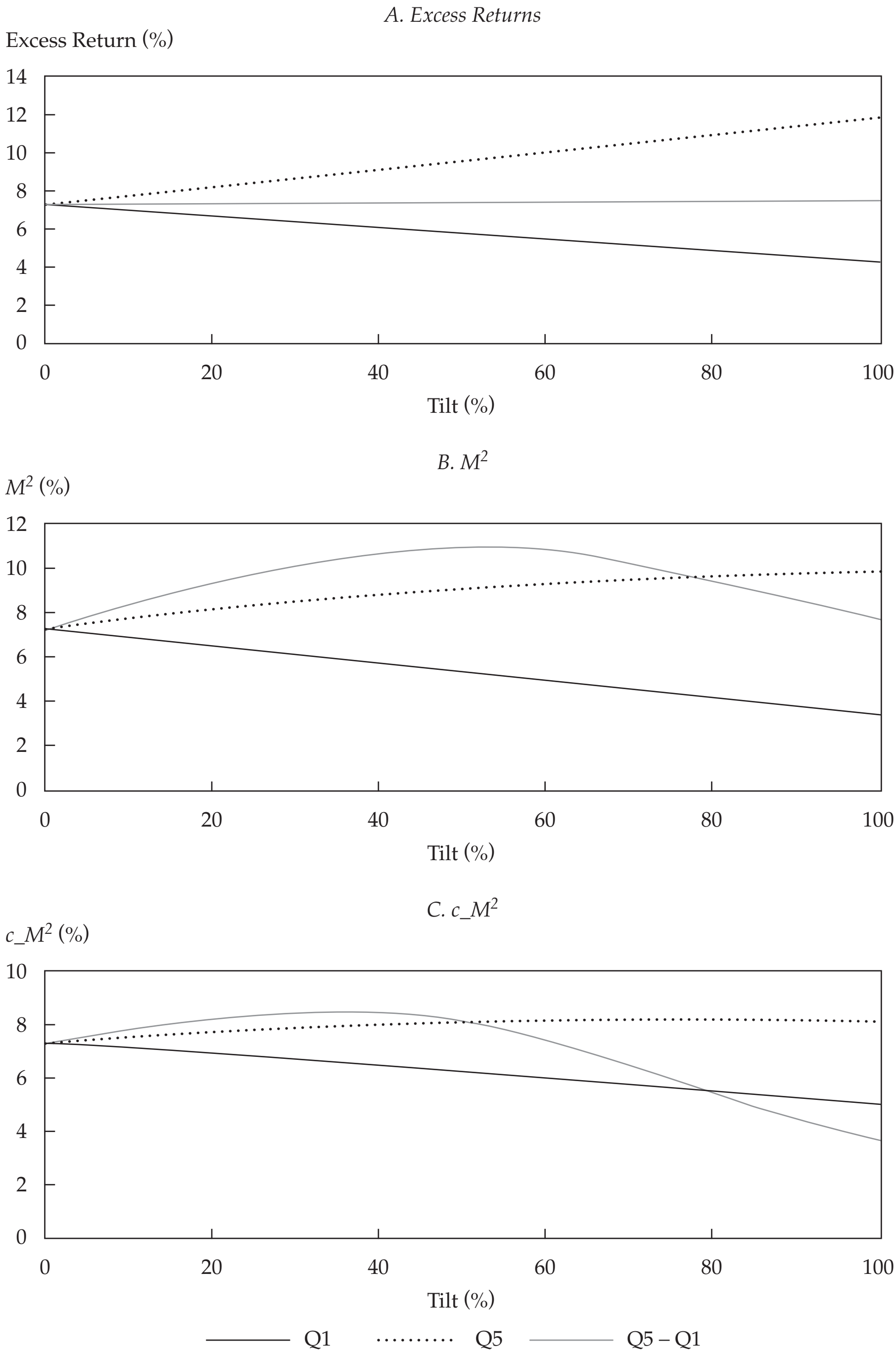
Note that this spread should be highly correlated with the much-heralded Fama–French HML (high minus low) BV/MV factor. Thus, an investment strategy that tries to mimic this factor by forming an optimal tilt with the market index appears to be dominated by other simple tilt strategies.

- *Tilting toward momentum quintiles.* A momentum investment strategy has been highly profitable historically, as Table 4 and **Figure 3** demonstrate. The worst-performing quintile (Q1) of momentum stocks earned an average excess return of only 4.3 percent compared with the market portfolio's 7.4 percent. This Q1 performance translated into a Jensen's alpha of –3.8 percent (standard error of 1.9 percent). Without our data screen, this "loser" quintile's alpha would have been even lower, –5.7 percent. The best-performing quintile portfolio (Q5) earned a 4.1 percent alpha (standard error of 2.0 percent). As with the BV/MV anomaly, the optimal position was to be fully invested in the Q5 stocks. When the alpha is cut in half, the optimal position is about 80 percent invested in the Q5 momentum stocks, although allocations from 60–100 percent yield similar performance characteristics.

In contrast to the Q5–Q1 spread for BV/MV, shorting the loser stocks and going long in the winner stocks, as Table 4 shows, results in even higher optimal Sharpe ratios and M^2s than only going long in the winners. The optimal weight is about one-half in the spread, with M^2 equal to 11 percent.[20] The improvement observed here follows from the fact that the loser portfolio's alpha is almost as large in magnitude as the winner's alpha. The resulting large spread alpha more than offsets the increased residual risk from investing in the spread, as reflected in the higher information ratio for the spread—45.3 percent as compared with 38.3 percent for the Q5 stocks. At very high levels

[20]The estimate of residual risk in Panel B (17.6 percent) is higher than the standard deviation in Panel A for 100 percent invested in the spread (17.4 percent) because of the degrees-of-freedom adjustments, one for total variance and two for residual variance.

Figure 3. Momentum Portfolio Tilts, July 1963–June 1999 Data



of investment in the momentum spread, the residual risk effect dominates and the performance ratios quickly deteriorate. This can be clearly seen in Panels B and C of Figure 3.

Summary. Our results on the benefits of tilting an investment portfolio toward stocks of extreme size, value versus growth, or momentum quintiles based on the 36 years of return data studied lead to several conclusions. First, given our price and market-cap filters, portfolio efficiency is not improved much by tilting portfolios toward extreme size quintiles. Combining the market portfolio with value (high-BV/MV) stocks or past winners (momentum stocks) results in significant increases in the Sharpe ratio and M^2. Even if an investor believes that only half of the past positive performance of the value and momentum strategies is expected in the future, substantial tilting is desirable.

The Bayesian Approach to Optimal Portfolio Formation

The preceding analysis used historical data to estimate the inputs to the portfolio optimization problem and provided results for a variety of tilt strategies. Even if investors believe that the portfolio parameters are (relatively) constant over time, however, they should consider the potential impact of estimation error on portfolio decisions. Unfortunately, traditional statistical analysis is not well suited to this task. The standard errors reported earlier can be used to derive confidence intervals for, say, alphas, but how should such observations be translated into an investment decision?

Intuition for the Bayesian Analysis. Intuitively, if an alpha is not estimated with much precision, the apparent abnormal return (positive alpha) is likely to be a result of chance and, therefore, may not be a good indication of what will happen in the future. In such a case, tilting less aggressively in the direction of the given anomaly would seem sensible, but the extent to which an investor should "discount" the historical evidence because of this uncertainty or estimation risk is not so clear.

A related issue is that an investor may, even before looking at the data, have some prior notion as to a plausible range of values for alpha. This prior belief could be based on observations of returns in earlier periods or in other countries, or it could be based on economic theory or a general view about the efficiency of financial markets and the relevance of such paradigms as the CAPM.[21] For example, the CAPM implies that alpha should be zero when the index is the true market portfolio of all assets.

[21] See Pastor for an interesting analysis of the role of a pricing model in forming beliefs. Also see Black and Litterman (1992) and related earlier work by Jobson, Korkie, and Ratti (1979) and Jorion (1985).

Whatever the source of our prior belief, suppose that we judge an annualized alpha bigger than 4 percent to be implausibly large. Yet, we obtain a regression estimate of 4.7 percent. In light of our prior belief, this estimate is clearly too high, but how much do we want to discount this estimate? Naturally, the extent to which we will want to lower or *shrink* the estimated value depends on the confidence we have in our initial belief versus the precision of the statistical estimate.

Bayesian analysis provides an appealing framework in which to formalize these ideas and incorporate them in an optimal portfolio decision. Academic work on portfolio optimization has increasingly applied Bayesian methods; the study by Pastor (2000) is the most closely related to the issues considered here. In Bayesian analysis, initial beliefs about return parameters are represented in terms of prior probability distributions. For convenience, one assumes normal distributions for priors, as well as returns. Using a basic law of conditional probability known as Bayes' rule, one combines the data with one's initial beliefs to form an updated *posterior* probability distribution that reflects the learning that has occurred from observing the data.

Implementing the Bayesian Analysis. For pedagogical purposes, we initially suppose that alpha is the only unknown parameter. Let α_0 be the prior expected value of alpha and $\sigma(\alpha_0)$ be the prior standard deviation. Say α_0 equals 0, the value implied if the market index is mean–variance efficient. If $\sigma(\alpha_0)$ equals 2 percent, then an alpha of 4 percent or more is a 2 standard deviation event with probability less than 0.023. Of course, the actual alpha is either greater than 4 percent or not, but this probability quantifies our subjective judgment that such large values are implausible.

Now, let $\hat{\alpha}$ denote the given estimate of alpha and $se(\hat{\alpha})$ denote its standard error. Say $\hat{\alpha}$ is 4.7 percent, as previously, and $se(\hat{\alpha})$ is 1.5 percent, the values observed earlier for the high-BV/MV quintile in Table 3. In this context, Bayes' rule implies that the posterior mean is a precision-weighted average of the estimate and the prior mean:

$$\alpha^* = \frac{\alpha_0[1/\text{var}(\alpha_0)] + \hat{\alpha}[1/\text{var}(\hat{\alpha})]}{[1/\text{var}(\alpha_0)] + [1/\text{var}(\hat{\alpha})]}, \tag{5}$$

where *precision* is technically defined as the reciprocal of variance. If the prior uncertainty is large relative to how informative the data are [that is, if $\text{var}(\alpha_0)$ is high in relation to $\text{var}(\hat{\alpha})$], Bayes' rule places most of the weight on the estimate, $\hat{\alpha}$. Alternatively, if we do not have much data or if the data are quite noisy, $\text{var}(\hat{\alpha})$ is large and α^* is closer to the prior mean α_0.

In our example,

$$\frac{1}{\mathrm{var}(\alpha_0)} + \frac{1}{\mathrm{var}(\hat{\alpha})} = \frac{1}{0.02^2} + \frac{1}{0.015^2}$$
$$= 2{,}500 + 4{,}444.44$$
$$= 6{,}944.44,$$

so

$$\alpha* = \frac{2{,}500}{6{,}944.44}0 + \frac{4{,}444}{6{,}944.44}0.047$$

Because the estimate here is a bit more precise than the prior mean, greater weight is placed on the estimate than on the prior. As a result, the posterior mean of 3 percent is closer to 4.7 percent than to 0.

Having considered the idea of shrinking an estimate toward a prior mean, now we examine the impact of parameter uncertainty on risk. We noted earlier that the optimal amount of tilting toward a quintile or spread portfolio depends on its residual risk as well as its alpha and the market index risk–return parameters. From a Bayesian perspective, uncertainty about the true value of alpha is naturally recognized as an additional source of risk that confronts an investor. Conventional risk measures, however, ignore this *estimation risk*. To convey this point, we rewrite Equation 3 as a Bayesian *predictive* regression:

$$R_q - R_f = \alpha* + \beta_q(R_m - R_f) + [\varepsilon_q + (\alpha_q - \alpha*)], \quad (6)$$

where $\alpha*$ is the posterior mean for alpha discussed previously.

For an investor looking into the future, uncertainty about the manner in which the true alpha deviates from its posterior expected value is a form of *residual* or nonmarket risk. Recall that the residual term ε_q reflects economic influences that affect the stocks in portfolio q but does not have a net impact on the market index. Similarly, whether our expected value for alpha is too high or too low will have no bearing on whether the market subsequently goes up or down. Therefore, $\alpha_q - \alpha*$ is correlated neither with the market return nor, by a similar argument, with ε_{qt}.

To make an optimal portfolio decision, we need to know how much additional residual variability is induced by the "parameter uncertainty" associated with alpha. Formally, we require the variance of the posterior probability distribution for alpha. Fortunately, a simple and intuitive mathematical result delivers this variance: The posterior precision is simply the sum of the precisions of the prior and the alpha estimate, as given by the denominator of Equation 5. Recalling our earlier computations, this denominator is 6,944.44, so the posterior standard deviation, $\sigma*(\alpha)$, is 0.012, or 1.2 percent, compared with the prior standard deviation of 2 percent. The reduction from 2 percent to 1.2 percent is an indication of the extent to which observing the historical data has narrowed our belief about the true value of alpha.

Once we have the posterior standard deviation for alpha, the next step is to quantify the overall ***predictive residual risk***, $\sigma*(res)$, perceived by the investor—the standard deviation of the quantity in brackets in Equation 6.[22] Because ε_{qt} is uncorrelated with $\alpha - \alpha*$, $\sigma*(res)$ is

$$(0.084^2 + 0.012^2)^{0.5} = 8.5\%.$$

Interestingly, the uncertainty about alpha increases the perceived residual risk by only 0.1 percentage point from the regression estimate of 8.4 percent for the high-BV/MV quintile. Although one might be inclined to attribute this to the fairly tight prior distribution assumed for alpha, that is not the cause. To see why, suppose the prior is totally uninformative; that is, let $\sigma(\alpha_0)$ approach infinity in Equation 5. Now, the posterior moments are identical to the sample moments:

$$\alpha* = \hat{\alpha} = 4.7\%$$

and

$$\sigma*(\alpha) = se(\hat{\alpha}) = 1.5\%.$$

The implied value of $\sigma*(res)$ increases only slightly, however, and is still about 8.5 percent. In this case, the investor's uncertainty is dominated by the variability of the residual component of return. Parameter uncertainty is a second-order effect.

We can make similar computations without the simplifying assumption that alpha is the only unknown parameter in the decision problem. The relevant formulas are in Appendix A. Having uninformative priors for alpha and beta but treating the sample residual variance as the true value of $\mathrm{var}(\varepsilon_q)$, we obtain 8.6 percent for $\sigma*(res)$. If we let the data dominate our belief about $\mathrm{var}(\varepsilon_q)$ and use uninformative priors for all the regression parameters, $\sigma*(res)$ increases to 8.9 percent.

To examine the impact of estimation risk on asset allocation, we combine the original estimate, $\hat{\alpha} = 4.7$ percent, with our most conservative estimate of residual risk, $\sigma*(res) = 8.9$ percent. This risk measure now takes on the role played earlier by the regression estimate of residual standard deviation. For

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[22] In Bayesian analysis, one refers to "posterior" uncertainty when talking about parameter values and "predictive" uncertainty when considering the future value of a random variable, such as a return whose distribution depends on the parameters. Both concepts involve beliefs formed after observing past data.

simplicity, we also specify uninformative priors for the mean and variance of the market index. By an argument similar to that for alpha and residual risk, uncertainty about the market's true mean return increases the (predictive) risk perceived by the investor and lowers the perceived market Sharpe ratio.

Other things being equal, this market effect tends to increase the optimal weight on the tilt portfolio. Recall from our earlier analysis of the BV/MV anomaly that when we ignore parameter uncertainty and assume no short selling, being fully invested in the high-BV/MV quintile is optimal. The corresponding M^2 value is 11.3 percent, compared with the market expected return of 7.4 percent (market standard deviation of 17.7 percent). With parameter uncertainty, being fully invested is still the optimal strategy: The predictive market risk is 18.5 percent, and the optimal M^2 is perceived to be 11.2 percent. In this case, the investment decision is not affected by ignoring parameter uncertainty.

A more interesting illustration is to suppose that we shrink the estimate of alpha with weights $c = 0.7$ on an alpha of 0 and 0.3 on $\hat{\alpha} = 4.7$ percent. The resulting value of alpha is 1.4 percent. Without incorporating parameter uncertainty, the optimal weight on the high-BV/MV quintile would be 74.6 percent, which is still quite high, despite the more conservative assumption about alpha. The Bayesian optimal weight is a bit lower, at 72.5 percent, because the increase in residual risk apparently dominates the market risk effect. Using the "wrong" weight, 74.6 percent, would reduce the perceived M^2 by less than 1 basis point from the optimal predictive value of 7.9 percent. Even in this case of an unconstrained optimum, neglecting estimation risk has virtually no effect on the investor. Only the desired degree of Bayesian shrinkage is important. Similar conclusions hold for tilts involving the small-company and high-momentum quintiles.[23]

Summary. Bayesian analysis is a simple and intuitive approach for incorporating information about the imprecision or uncertainty in the historical estimates of alpha or beta. This uncertainty increases the perceived (or predictive) residual risk of the investment portfolio. If this effect is greater than the effect of uncertainty about the market expected return, the investor should be inclined to tilt the portfolio less aggressively toward the anomaly strategy.

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[23] Shrinkage implies that the prior for alpha is informative, which means that our analysis with uninformative priors overstates the impact of estimation risk in this case. Also, our conclusions are essentially unchanged if parameter uncertainty regarding the market index parameters is ignored.

The discussion in this section formalized these concepts in the context of optimal portfolio formation. We found that, under plausible assumptions, giving consideration to parameter uncertainty changes the optimal asset allocation to some extent but not substantially. Shrinkage of alpha toward a prior mean is the dominant aspect of the Bayesian analysis in this application.

Additional Portfolio Results

In this section, we consider (1) the characteristics and performance of our tilt portfolios over a two-year horizon, (2) sensitivity of one-year results to using a portfolio of both bonds and stocks as the market index, and (3) the formation of optimal portfolios from stocks only but encompassing all three anomaly strategies simultaneously.

Tilt Portfolios over a Two-Year Horizon. The previous analysis examined portfolio performance for the one year immediately following the formation of the size, BV/MV, and momentum portfolios. For an investor with a longer horizon, performance measurement for one year implicitly assumes that the optimal portfolio will be rebalanced every year. Rebalancing, however, may entail considerable transaction costs and is warranted only if the performance of the portfolio is likely to decay substantially over time. If the relevant characteristics of stocks do not change much for a one-year holding period, the investor can anticipate similar portfolio results if the stocks are held for an additional year. This assumption seems plausible for stocks ranked by size and BV/MV but not for those ranked by momentum, which by its nature is relatively short-lived.

■ *Portfolio characteristics.* **Table 5** reports the *transition probabilities* for stocks moving from one quintile portfolio to another in one year. (The underlying data are the same as those used in optimal portfolio construction.) For example, the second row of Panel A in Table 5 shows that a stock in size Quintile 1 (the smallest-cap stocks) has a 43.9 percent probability of being in the same size quintile in the following year. The corresponding probability for the lowest-BV/MV quintile is 54.4 percent, and for the loser quintile of momentum stocks, it is 20.7 percent. The probability of transition to the "Missing" cell is quite high for stocks in all quintiles and especially so for the stocks in the first quintile.

Generally, the transition probabilities for stocks ranked on momentum are roughly the same, regardless of the initial quintile—which is not surprising. Momentum is an indication of persistence, in the sense that high (low) previous-year quintile returns are associated with high (low) average or expected returns for the following year. However, the realized returns in any holding period will, like returns generally, be dominated by surprises that

Table 5. One-Year-Ahead Transition Probabilities for Stocks in a Given Quintile, July 1963–June 1999 Data

Old/New Quintile	Size	BV/MV	Momentum
A. Move in year t + *1 from Quintile 1 in year* t *to:*			
Missing	34.6%	18.3%	19.4%
Q1	43.9	54.4	20.7
Q2	18.7	20.5	16.4
Q3	2.6	4.7	14.5
Q4	0.2	1.6	14.7
Q5	0.0	0.5	14.2
B. Move in year t + *1 from Quintile 2 in year* t *to:*			
Missing	14.9%	14.9%	13.5%
Q1	17.3	12.3	16.1
Q2	48.3	43.2	20.9
Q3	17.9	22.2	20.3
Q4	1.6	5.9	17.2
Q5	0.1	1.5	11.9
C. Move in year t + *1 from Quintile 3 in year* t *to:*			
Missing	10.8%	14.4%	12.3%
Q1	1.9	1.9	14.7
Q2	15.0	16.6	20.3
Q3	56.6	39.5	21.8
Q4	15.5	22.9	19.0
Q5	0.2	4.7	11.9
D. Move in year t + *1 from Quintile 4 in year* t *to:*			
Missing	9.3%	13.9%	12.8%
Q1	0.2	0.5	16.1
Q2	0.8	3.7	18.4
Q3	11.6	18.8	19.7
Q4	68.6	42.7	19.3
Q5	9.5	20.4	13.8
E. Move in year t + *1 from Quintile 5 in year* t *to:*			
Missing	7.4%	16.0%	15.7%
Q1	0.0	0.3	20.6
Q2	0.0	0.8	14.9
Q3	0.1	3.5	13.6
Q4	6.2	16.8	16.6
Q5	86.2	62.6	18.7

Note: A stock was considered "missing" in year $t + 1$ if it was delisted (because of mergers, acquisitions, delistings, and bankruptcies) or if it did not meet the investment criteria we used (a price greater than $2 and size above the lowest decile of market cap for NYSE stocks).

represent deviations from the expected returns. Thus, conditioning on the previous year's return provides limited information about next year's return and a given stock is about as likely to be in one future return (momentum) quintile as any other.

The transition probabilities in Table 5 indicate that a nontrivial fraction of the stocks in all the extreme quintiles, except the large-cap quintile, end up in another quintile after one year. To help in understanding the significance of these transition probabilities, consider the degree to which company characteristics tend to change over time. **Table 6** reports simple averages of company size, BV/MV, and momentum for each quintile portfolio in the formation (ranking) year, year *t*, and the five following years. In calculating the average market values, we included all the stocks in year *t* that survived in the successive future years. That is, we did not drop stocks that failed to continue to meet our initial price and size criteria.

Table 6 shows the average market cap of the small-cap stocks in Q1 rising steadily from $43.3 million in the formation year to $114.1 million in year *t* + 5.

Table 6. Characteristics of Stocks through Event Time, July 1963–June 1999 Data

Quintile in Year *t*	Year *t*	Year *t* + 1	Year *t* + 2	Year *t* + 3	Year *t* + 4	Year *t* + 5
A. Future market value of stocks in a given size quintile in year t *(millions)*						
Q1	$ 43	$ 57	$ 69	$ 85	$ 102	$ 114
Q2	77	89	103	118	133	152
Q3	148	164	182	205	227	257
Q4	356	390	425	470	519	571
Q5	2,995	3,167	3,325	3,503	3,692	3,885
B. Future BV/MV value of stocks in a given BV/MV quintile in year t						
Q1	0.20	0.29	0.36	0.41	0.47	0.50
Q2	0.44	0.51	0.57	0.61	0.66	0.70
Q3	0.66	0.72	0.76	0.80	0.82	0.85
Q4	0.92	0.95	0.97	0.98	1.00	1.00
Q5	3.00	2.90	2.80	2.75	2.74	2.65
C. Future momentum of stocks in a given momentum quintile in year t						
Q1	–24.2%	15.7%	20.2%	20.7%	20.5%	19.8%
Q2	–0.6	15.1	17.0	16.5	17.7	17.8
Q3	14.0	16.2	16.4	16.9	16.4	16.9
Q4	32.4	17.4	16.6	17.6	17.1	16.6
Q5	92.3	21.8	16.9	17.5	17.0	17.3

The Q5 stocks' average market value increases more modestly, in percentage terms—from almost $3 billion in year *t* to $3.8 billion in year *t* + 5. The value stocks (i.e., stocks in the highest-BV/MV quintile, Q5) experienced a slight and steady decrease in BV/MV during the five-year period, with the average BV/MV declining from 3.00 to 2.65 in year *t* + 5. Finally, although there is substantial reversion toward the mean in year *t* + 1, the first-year momentum effect is apparent in the (simple) average returns. The losers (Q1 stocks) underperform winners (Q5 stocks) by about 6 percentage points. This advantage is reversed in years *t* + 2 and beyond, with the original (year *t*) losers outperforming the winners by 3 percentage points or so each year. On the one hand, the results in Table 6 indicate that if a tilt portfolio were not rebalanced at the end of one year, its average market cap and BV/MV would not differ much from those of a portfolio that was reconstructed every year. On the other hand, the transition probabilities in Table 5 suggest that the dispersion of these characteristics may increase somewhat over time. Ultimately, the performance of the anomaly-based strategies over longer holding periods is an empirical question.

■ *Portfolio performance.* We now examine performance of the tilt portfolios in the second year following construction. The question is whether the expected gains from tilting in the first year are sustained in the second year without rebalancing.

To implement a strategy of investing in the second year, we sorted stocks on company characteristics at the end of June (May, for momentum) of year *t* but measured the tilt portfolio returns (using the companies that survived) starting in July of year *t* + 1. The sample spans 35 years and starts in July 1964.

The performance results reported in **Table 7** and depicted in **Figure 4** indicate that, although Quintile 1 now has a positive alpha, tilting toward small companies in the second year provides little improvement in the risk–return trade-off.

As shown in **Table 8** and **Figure 5**, the value stocks (Q5 in the BV/MV group) put forth a strong risk-adjusted performance in the second year after their formation. They earned an average excess return of 10 percent, compared with the value-weighted market return of 7.1 percent. With a beta of just 0.78, the value stocks' Jensen's alpha is 4.5 percent (standard error of 1.1 percent), similar to the 4.7 percent alpha they earned for the first year.[24] The information ratio actually increased in the second year (from the 55.9 percent of year *t* + 1 to 72.7 percent in year *t* + 2) because residual risk declined substantially—from 8.4 percent to 6.2 percent. Because the *c_Sharpe*

[24]These results are consistent with those of La Porta, Lakonishok, Shleifer, and Vishny (1997).

Table 7. Second-Year Performance of Portfolios Tilted toward Size Quintiles, July 1963–June 1999 Data

A. Tilt portfolios' performance

Variable	0%	20%	40%	50%	60%	80%	100%
Quintile 1							
exrt	7.1%	7.6%	8.2%	8.5%	8.7%	9.3%	9.8%
σ(*exrt*)	17.9	18.8	20.1	20.9	21.8	23.7	25.9
α	0.0	0.3	0.6	0.7	0.9	1.1	1.4
Sharpe	39.7	40.7	40.7	40.5	40.1	39.0	37.9
c_Sharpe	39.7	39.9	39.3	38.7	38.1	36.6	35.1
M^2	7.1	7.3	7.3	7.2	7.2	7.0	6.8
c_M^2	7.1	7.1	7.0	6.9	6.8	6.6	6.3
Quintile 5							
exrt	7.1%	7.0%	6.9%	6.9%	6.8%	6.7%	6.7%
σ(*exrt*)	17.9	17.7	17.5	17.5	17.4	17.3	17.2
α	0.0	0.0	0.0	0.0	0.0	0.0	–0.1
Sharpe	39.7	39.6	39.5	39.4	39.3	39.0	38.7
c_Sharpe	39.7	39.6	39.5	39.4	39.4	39.1	38.9
M^2	7.1	7.1	7.1	7.0	7.0	7.0	6.9
c_M^2	7.1	7.1	7.1	7.1	7.0	7.0	7.0
Quintile 5 – Quintile 1							
exrt	7.1%	5.0%	3.0%	2.0%	0.9%	–1.1%	–3.2%
σ(*exrt*)	17.9	13.9	11.5	11.2	11.6	14.2	18.2
α	0.0	–0.3	–0.6	–0.7	–0.9	–1.2	–1.5
Sharpe	39.7	36.2	26.1	17.6	8.1	–7.8	–17.4
c_Sharpe	39.7	37.3	28.7	20.9	12.0	–3.6	–13.3
M^2	7.1	6.5	4.7	3.2	1.5	–1.4	–3.1
c_M^2	7.1	6.7	5.1	3.7	2.1	–0.6	–2.4

B. 100% tilt and optimal portfolio

Variable	Q1	Q5	Q5 – Q1
α (%)	1.4	–0.1	–1.5
	(2.8)	(0.6)	(3.3)
β	1.18	0.95	–0.23
	(0.15)	(0.03)	(0.17)
σ(ε) (%)	15.3	3.1	18.0
Shp_mkt (%)	39.7	39.7	39.7
	(16.9)	(16.9)	(16.9)
Info (%)	9.4	–1.9	–8.3
	(18.2)	(18.2)	(18.2)
Shp_opt (%)	40.0	39.7	39.9
	(16.9)	(16.9)	(16.9)

Note: See Table 2 notes.

Figure 4. Second-Year Size-Tilt Portfolio Performance, July 1963–June 1999 Data

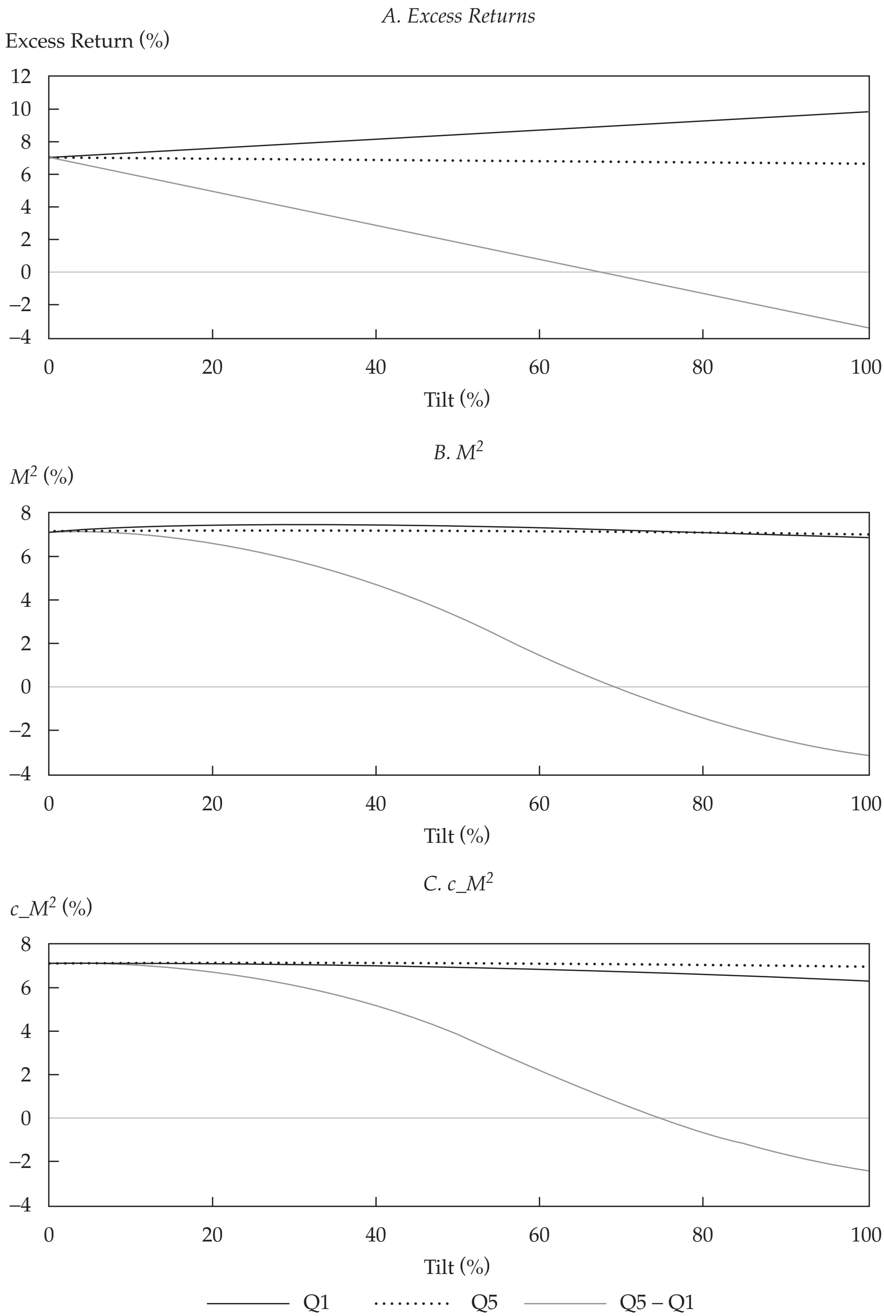


Table 8. Second-Year Performance of Portfolios Tilted toward BV/MV Quintiles, July 1963–June 1999 Data

A. Tilt portfolios' performance

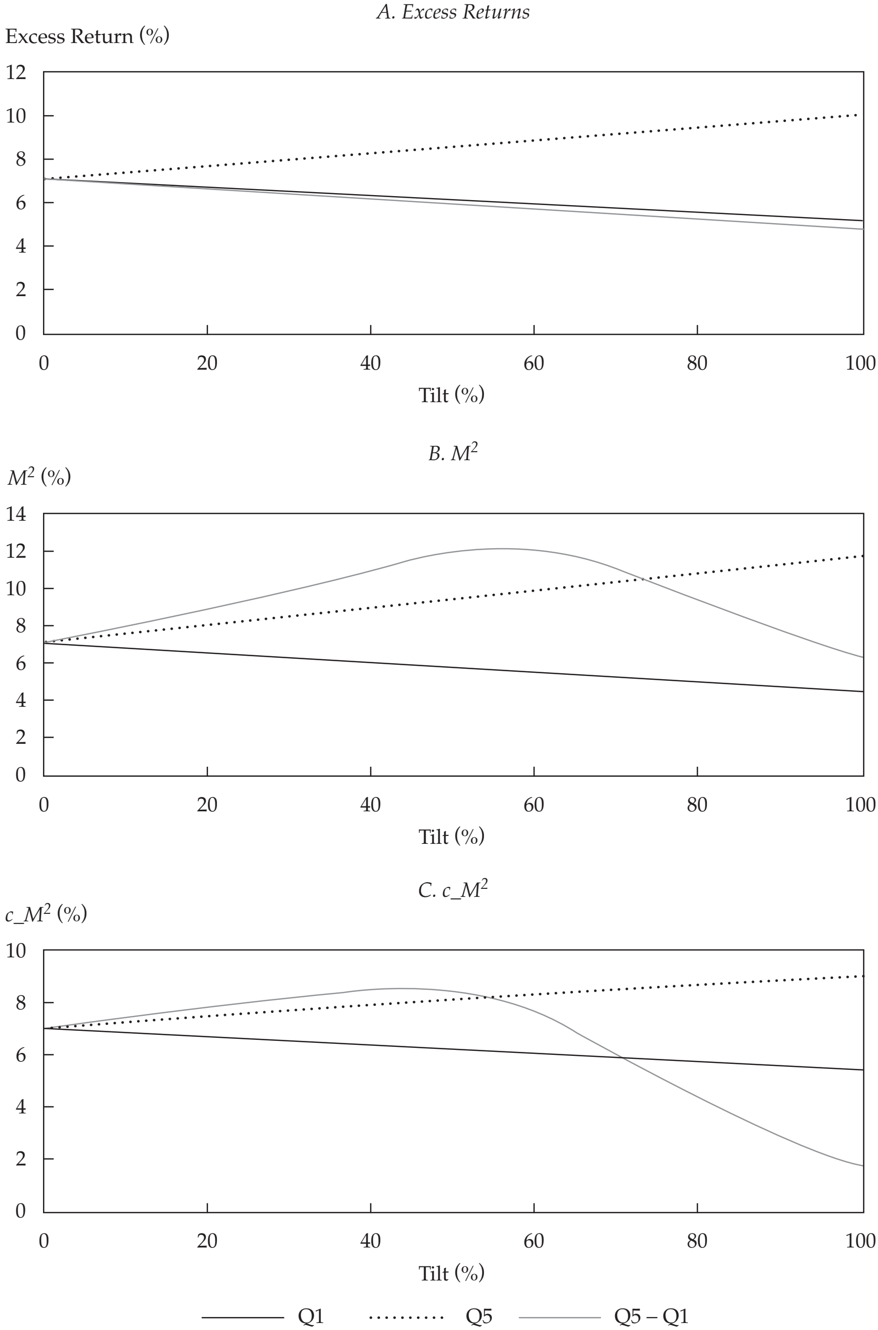
Variable	0%	20%	40%	50%	60%	80%	100%
Quintile 1							
exrt	7.1%	6.7%	6.3%	6.2%	6.0%	5.6%	5.2%
σ(*exrt*)	17.9	18.2	18.7	19.0	19.3	20.0	20.9
α	0.0	–0.5	–1.0	–1.2	–1.5	–1.9	–2.4
Sharpe	39.7	36.9	33.9	32.4	30.9	27.9	25.0
c_Sharpe	39.7	38.2	36.5	35.6	34.7	32.8	30.8
M^2	7.1	6.6	6.1	5.8	5.5	5.0	4.5
c_M^2	7.1	6.8	6.5	6.4	6.2	5.9	5.5
Quintile 5							
exrt	7.1%	7.7%	8.3%	8.6%	8.8%	9.4%	10.0%
σ(*exrt*)	17.9	17.1	16.5	16.2	15.9	15.5	15.2
α	0.0	0.9	1.8	2.2	2.7	3.6	4.5
Sharpe	39.7	44.8	50.1	52.8	55.5	60.8	65.9
c_Sharpe	39.7	42.2	44.7	45.9	47.1	49.3	51.1
M^2	7.1	8.0	9.0	9.5	9.9	10.9	11.8
c_M^2	7.1	7.6	8.0	8.2	8.4	8.8	9.2
Quintile 5 – Quintile 1							
exrt	7.1%	6.6%	6.2%	5.9%	5.7%	5.2%	4.8%
σ(*exrt*)	17.9	13.5	9.9	8.8	8.4	10.0	13.5
α	0.0	1.4	2.8	3.5	4.1	5.5	6.9
Sharpe	39.7	49.2	62.1	67.2	67.6	52.6	35.3
c_Sharpe	39.7	44.1	48.2	47.7	43.1	24.9	9.8
M^2	7.1	8.8	11.1	12.0	12.1	9.4	6.3
c_M^2	7.1	7.9	8.6	8.5	7.7	4.5	1.8

B. 100% tilt and optimal portfolio

Variable	Q1	Q5	Q5 – Q1
α (%)	–2.4	4.5	6.9
	(1.5)	(1.1)	(2.3)
β	1.08	0.78	–0.30
	(0.08)	(0.06)	(0.12)
σ(ε) (%)	8.1	6.2	12.6
Shp_mkt (%)	39.7	39.7	39.7
	(16.9)	(16.9)	(16.9)
Info (%)	29.8	72.7	54.7
	(18.2)	(18.2)	(18.2)
Shp_opt (%)	42.4	53.8	48.2
	(16.9)	(16.9)	(16.9)

Note: See Table 2 notes.

Figure 5. Second-Year BV/MV-Tilt Portfolio Performance, July 1963–June 1999 Data



and c_M^2 measures are maximized at the 100 percent tilt, the optimal portfolio would be invested entirely in value stocks while shorting the market portfolio. Realistically, the implied strategy is to invest *primarily* in value stocks.

Consistent with Table 6, the momentum strategy's performance in the second year, as shown in **Table 9** and **Figure 6**, exhibits signs of reversal, with an alpha for the spread portfolio of –4.1 percent (standard error of 2.5 percent). Reversal in the second year is consistent with momentum in the first year being a continuation of overreaction to information that arrived during the portfolio formation year rather than an adjustment to an initial under-reaction to information.

Results with a Market Portfolio of Bonds and Stocks. Financial planners typically recommend portfolios that consist of substantial investments in equity and bond securities. A mix of approximately 60 percent equities and 40 percent bonds is a common recommendation. Therefore, an interesting question is whether and by how much one should deviate from such a balanced market portfolio in light of the size, BV/MV, and momentum anomalies discussed earlier.

We constructed a time series of returns to a stock and bond market portfolio by combining the CRSP value-weighted stock portfolio returns with 10-year U.S. T-bond returns. Except for this different market index, we then repeated our empirical analysis of the benefits of tilting toward size, BV/MV, and momentum quintiles. These results are not shown in tables and figures (which are available from the authors on request) but the primary findings are as follows.

The combined index for the 60/40 portfolio had a lower average excess return (5.1 percent) and lower risk (13.5 percent) than the all-stock index used in previous tables (7.4 percent return and 17.7 percent standard deviation). The index Sharpe ratio declined slightly with the inclusion of bonds, from 0.42 to about 0.38, and the alphas of the extreme quintiles relative to the 60/40 market index were slightly higher than they were relative to the all-stock index. Given the lower volatility of the 60/40 mix, the betas of the quintiles all naturally increased.[25]

[25]To gain intuition for this increase in beta, assume the bond return is riskless; then, let $\beta_i = \mathrm{cov}(R_i, R_m)/\mathrm{var}(R_m)$ be the beta of security i with respect to the equity index return, R_m. Because R_f does not covary with stock returns and its variance is assumed to be zero, the beta of security i with respect to the 60/40 blend of R_m and the riskless bond is $\mathrm{cov}(R_i, 0.6R_m + 0.4R_f)/\mathrm{var}(0.6R_m + 0.4R_f) = 0.6\mathrm{cov}(R_i, R_m)/0.36\,\mathrm{var}(R_m) = 1.67\beta_i$. The increase we observed in the data reflects the relatively low variability of bond returns as compared with stock returns.

Table 9. Second-Year Performance of Portfolios Tilted toward Momentum Quintiles, July 1963–June 1999 Data

A. Tilt portfolios' performance

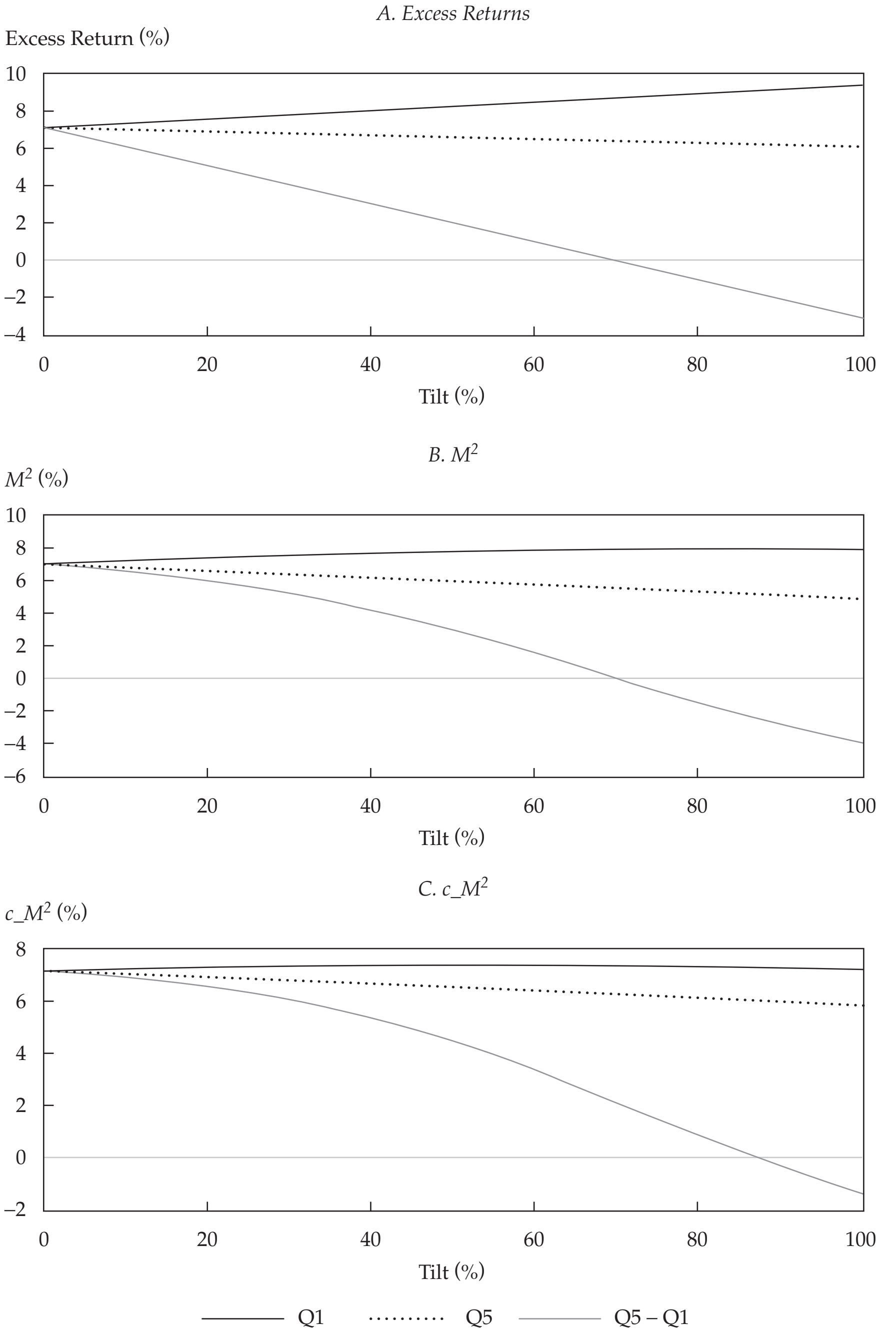
Variable	0%	20%	40%	50%	60%	80%	100%
Quintile 1							
exrt	7.1%	7.5%	8.0%	8.2%	8.4%	8.8%	9.2%
σ(*exrt*)	17.9	18.1	18.5	18.8	19.1	19.8	20.7
α	0.0	0.4	0.8	1.0	1.1	1.5	1.9
Sharpe	39.7	41.6	43.0	43.5	43.9	44.4	44.6
c_Sharpe	39.7	40.5	40.9	41.0	40.9	40.6	40.0
M^2	7.1	7.4	7.7	7.8	7.9	7.9	8.0
c_M^2	7.1	7.3	7.3	7.3	7.3	7.3	7.1
Quintile 5							
exrt	7.1%	6.9%	6.7%	6.6%	6.5%	6.3%	6.1%
σ(*exrt*)	17.9	18.6	19.4	19.8	20.2	21.2	22.2
α	0.0	–0.4	–0.9	–1.1	–1.3	–1.8	–2.2
Sharpe	39.7	37.2	34.6	33.4	32.1	29.7	27.4
c_Sharpe	39.7	38.4	36.9	36.2	35.4	33.9	32.4
M^2	7.1	6.7	6.2	6.0	5.8	5.3	4.9
c_M^2	7.1	6.9	6.6	6.5	6.3	6.1	5.8
Quintile 5 – Quintile 1							
exrt	7.1%	5.1%	3.0%	2.0%	1.0%	–1.1%	–3.1%
σ(*exrt*)	17.9	15.1	13.0	12.3	11.9	12.3	13.9
α	0.0	–0.8	–1.6	–2.1	–2.5	–3.3	–4.1
Sharpe	39.7	33.5	23.2	16.2	8.1	–8.8	–22.4
c_Sharpe	39.7	36.3	29.6	24.6	18.4	4.6	–7.6
M^2	7.1	6.0	4.2	2.9	1.4	–1.6	–4.0
c_M^2	7.1	6.5	5.3	4.4	3.3	0.8	–1.4

B. 100% tilt and optimal portfolio

Variable	Q1	Q5	Q5 – Q1
α (%)	1.9	–2.2	–4.1
	(1.7)	(1.4)	(2.5)
β	1.03	1.17	0.14
	(0.09)	(0.07)	(0.13)
σ(ε) (%)	9.6	7.5	13.9
Shp_mkt (%)	39.7	39.7	39.7
	(16.9)	(16.9)	(16.9)
Info (%)	19.9	–29.5	–29.6
	(18.2)	(18.2)	(18.2)
Shp_opt (%)	40.0	42.3	42.4
	(16.9)	(16.9)	(16.9)

Note: See Table 2 notes.

Figure 6. Second-Year Momentum-Tilt Portfolio Performance, July 1963–June 1999 Data



The changes in residual risk are less consistent than the changes in alphas and betas. Most noteworthy are increases from 9.9 percent to 12.8 percent for growth stocks (Q1 as measured by BV/MV) and from 10.3 percent to 12.3 percent for past losers (Q1 as measured by momentum). Despite these changes, the implications for tilting and optimal asset allocation are quite similar to those discussed earlier for the all-stock portfolios: Optimal strategies call for heavy tilts toward value and high-momentum stocks, with less aggressive tilts toward the value and momentum spreads because of the moderating effect of residual risk.

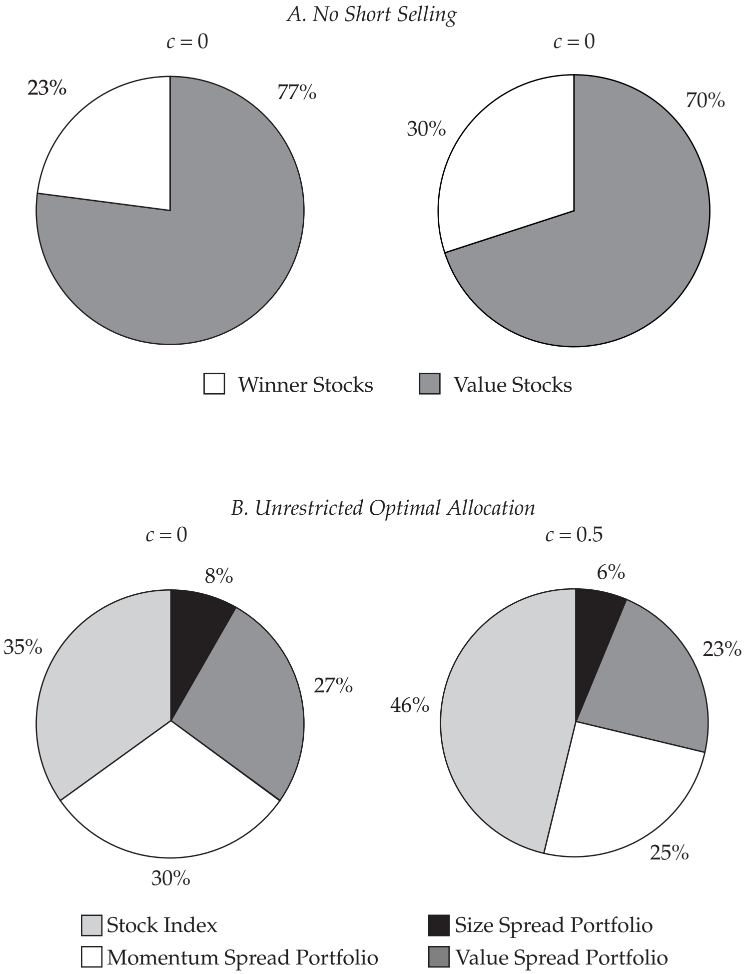
Using All Three Anomaly Strategies. In this section, we consider optimal portfolio formation using two sets of risky assets: (1) the stock market index, large companies, value companies, and high-momentum companies and (2) the stock market index, the size spread (large – small), the value spread (high BV/MV – low BV/MV), and the momentum spread (winners – losers). The individual risk and return characteristics of these assets were examined in the "Optimal Portfolio Tilts" section.

Also relevant to the joint optimization problem are the correlations between portfolio residual returns. The estimated residual correlation between large companies and value companies is –0.30. The residual correlation between the BV/MV and momentum (size) spreads is –0.27 (–0.20). The other correlations are also negative but closer to zero.

Following Treynor and Black, we structured the optimal portfolio decision in terms of an optimal active portfolio of the anomaly-based investments and an optimal combination of the market index and the active portfolio.[26] Based on the historical estimates, the optimal strategy for the first set of assets entails an "unreasonably" large (–563 percent) short position in the market index. In fact, even if we let $c = 0.9$ and reduce all the alphas by 90 percent, the optimal strategy still shorts the market. As Panel A of **Figure 7** shows, when short selling is ruled out, the active portfolio with $c = 0$ consists of 77 percent in value stocks and 23 percent in winner stocks and the optimal portfolio has no (direct) investment in the stock market index. The M^2 of this strategy is 11.5 percent, much higher than the 7.4 percent excess return on the market. Note that the greater investment in value stocks, as compared with winner momentum stocks, is consistent with the higher information ratio for those stocks that we noted earlier. Letting $c = 0.5$ reduces the weight on value stocks to 70 percent,

[26] In the unrestricted-short-selling case, we used the formulas in Gibbons, Ross, and Shanken (1989), which generalize those in Treynor and Black to accommodate nonzero residual correlations.

Figure 7. Optimal Allocation with Three Anomaly-Based Strategies



with 30 percent now in winner stocks. The M^2 drops to 9.2 percent (keep in mind that M^2 is M to the power of 2).

Finally, if we let $c = 0.75$ (not shown), reflecting less confidence in the historical alphas, the active portfolio would consist of 8 percent in large-cap stocks, 60 percent in value stocks, and 32 percent in winner stocks. The resulting portfolio still would have no investment in the market index, and the optimal M^2 would drop to 8.0 percent. Not much changes when we assume that the large-stock alpha is zero. The small optimal investment in large stocks when $c = 0.75$ is driven by the diversification benefit of the negative residual correlations with value and, to a lesser extent, winner stocks. This diversification benefit becomes relevant when the cost, in terms of lost expected return from investment in value and winner stocks, is reduced sufficiently (high c). The relative robustness of optimal portfolio weights to substantial reductions in forward-looking alphas is, we think, interesting and somewhat surprising.

Now consider investment in our second set of assets—the stock index and the spread portfolios. Recall that the "asset" in the case of a spread is a position consisting of \$1 in T-bills and \$1 on each side of the spread. The unrestricted optimal allocation with $c = 0$ looks more reasonable in this case than it did for the first set of assets. The active portfolio would have 12 percent invested in the size spread, 42 percent in the value spread, and 46 percent in the momentum spread. The optimal portfolio puts 35 percent in the stock index and 65 percent in the active portfolio. Panel B of Figure 7 shows how this allocation translates into an overall portfolio composition. The optimal portfolio has an M^2 of 13.9 percent, higher than the 11.5 percent for the first set of assets. The high residual risks of the individual spread positions are substantially reduced by diversifying across spreads, making it possible to exploit the high alphas more efficiently than with the single-spread tilts examined in the "Optimal Portfolio Tilts" section.

In general, when short selling is not restricted, we can show algebraically that increasing c leaves the active portfolio unchanged, although, naturally, the weight on the active portfolio is lowered. With $c = 0.5$ (0.75), that weight is 54 percent (40 percent), down from 65 percent, and the corresponding M^2 drops to 9.4 percent (7.9 percent) from the 13.9 percent value when $c = 0$. The unrestricted allocations in an overall portfolio for $c = 0.5$ are displayed in Panel B of Figure 7. Although the role of the anomaly-based strategies is reduced, we still see significant investment in these strategies even after substantial reductions in the historical alphas.

Conclusions and Directions for Future Research

Our main findings are as follows. First, our data show essentially no size effect in the U.S. equity market during the 1963–99 period. This absence of a size effect is a result, in part, of our exclusion of very-small-cap and low-priced stocks in an attempt to approximate realistic investment strategies. As in earlier work, the BV/MV and momentum effects are large. When we ranked stocks from low to high and formed BV/MV and momentum quintiles, the spreads in alpha between Q5 and Q1 were 5.4 percentage points for the BV/MV quintiles and 8.0 percentage points for the momentum quintiles. Considering the anomalies separately and examining feasible strategies involving either BV/MV or momentum stocks, the investor's optimal allocation is to be fully invested in Q5—high BV/MV or strong momentum. Moreover, this optimal allocation held true for the value stocks even if we injected a healthy dose of conservatism and reduced the alphas by half! The optimal tilt toward strong-momentum stocks with the reduced alpha was about 80 percent.

We obtained less extreme optimal tilts when we considered portfolios based on the Q5 – Q1 spreads (i.e., long in Q5 and short in Q1), but strategies based on these spreads are less relevant from a practical investment perspective. Interestingly, the residual risk of the value spread portfolio was so high that an investor would be better off with an aggressive position in high-BV/MV stocks than with an optimal spread position. Thus, despite the higher alpha of the spread portfolio (i.e., our version of the Fama–French HML factor portfolio), its risk–return characteristics are not as attractive as those of the high-BV/MV portfolio when considered solely in combination with the value-weighted market portfolio. The tenor of our results was unchanged when we introduced a 60/40 market index of stocks and bonds.

We also tracked the performance of each strategy in the second year after portfolio formation to provide some indication of the extent to which portfolio rebalancing is warranted. The value strategy of investing in high-BV/MV stocks continued to deliver strong abnormal performance in the second year, but the high-momentum stock alpha turned negative, which suggests the possibility of continuing investor overreaction as the source of momentum profits.

When we optimized with the market index, large-cap stocks, value stocks, and winner stocks, the optimal portfolio was about three-quarters in value and a quarter in momentum, even if we reduced the alphas by half. This optimization called for no direct investment in the market index. Using the size, value, and momentum spreads instead produced the highest performance measure for all the scenarios we considered—an M^2 of 13.9 percent as compared with the market excess return of 7.4 percent. The optimal allocation was about one-third in the value-weighted index, about 30 percent each for the value and momentum spreads, and the rest in the size spread. With alphas cut in half, the M^2 dropped to 9.4 percent. Almost half of the optimal portfolio was then invested in the market index, with about a quarter each in the value and momentum spreads.

That value and momentum should play an important role in optimal portfolio decisions is to be expected, given the literature on CAPM anomalies. The extent to which aggressive investment in these anomalies seems to be called for, even with substantial reductions in alphas and the incorporation of Bayesian estimation risk, is more unexpected.

We hope to have provided insights into the risk and return characteristics of anomaly-based investment strategies that will be useful to investors in making future portfolio decisions. An interesting future research pursuit would be an expansion of the analysis to include international investment opportunities as well as a consideration of tax effects and costs of anomaly-based investing. Incorporating the extensive literature on stock return predictability that documents changes in risk and expected return over time might also lead to improved optimal portfolio decisions.

Appendix A. More-General Bayesian Analysis

We now explore the potential impact of parameter uncertainty on predictive residual risk when beta and alpha are unknown. As earlier, returns are assumed to be jointly normally distributed. The priors for alpha and beta are taken to be uninformative. Because variances are estimated far more precisely than expected return parameters such as alpha, one might, in practice, reasonably have a strong prior about residual variance based on other data. For simplicity, we make the stronger assumption that the residual standard deviation is known. In this case, the posterior distribution of the regression parameters mirrors the usual sampling distribution for the regression estimates [i.e., (α, β) is jointly normally distributed with mean $(\hat{\alpha}, \hat{\beta})$ and variance matrix $\boldsymbol{\sigma}^2(\boldsymbol{\epsilon})(\mathbf{X}'\mathbf{X})^{-1}$], where $\mathbf{X}$ is the $T \times 2$ matrix of independent variables, including a constant vector (Zellner 1971).

Next, we consider the regression equation of quintile excess return on the market index excess return. We use the simple notation $y_t = \alpha + \beta x_t + \varepsilon_t$. For out-of-sample return observations x and y, the corresponding *predictive regression* is

$$y = \hat{\alpha} + \hat{\beta}x + [\boldsymbol{\epsilon} + (\alpha - \hat{\alpha}) + (\beta - \hat{\beta})x], \tag{A1}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are regression estimates based on data from $t = 1$ to T. From the Bayesian perspective, these estimates are the predictive regression coefficients ($\alpha*$ and $\beta*$ in the notation used in the section "Implementing Bayesian Analysis") and the expression in brackets is the predictive regression residual. The residual consists of the regression disturbance plus additional terms that reflect uncertainty about the parameters α and β.

The predictive residual can be viewed as the difference between y and the regression-based prediction of y conditional on a known value of x. This is referred to as the *prediction error* in standard regression analysis. Its conditional variance is well known and given by the following expression:

$$\text{var}(\boldsymbol{\epsilon})\left\{1 + \left[\frac{1 + (x - \bar{x})^2 / s_x^2}{T}\right]\right\}, \tag{A2}$$

where $\bar{x}$ is the sample mean and s_x^2 is the variance of the market returns (maximum-likelihood estimates).

Whereas x is known in the classical prediction problem, the future market return is yet to be realized when making the asset-allocation decision. Therefore, the relevant predictive residual variance for quintile return y is the average value of the variance in Expression A2 for all possible values of x. In this context, the regression estimates and the sample moments, $\bar{x}$ and s_x^2, are known and hence treated as nonrandom. With an uninformative prior for the market return parameters, the predictive mean is $\bar{x}$ and the predictive variance for x is $s_x^2\,(T+1)/(T-3)$, which is slightly larger than the sample variance (Zellner 1971). Taking the expectation of Equation A2 with respect to this distribution for x, we find that the predictive residual variance is

$$\text{var}^*(res) = \text{var}(\boldsymbol{\epsilon})\left[1 + \frac{2(T-1)}{T(T-3)}\right]. \tag{A3}$$

Interestingly, the influence of x has dropped out, simplifying the final result.

With an uninformative prior for $\text{var}(\boldsymbol{\epsilon})$, it can be shown that $\text{var}(\boldsymbol{\epsilon})$ is replaced in Expressions A2 and A3 by the usual unbiased residual variance estimate multiplied by $(T-2)/(T-4)$.

These formulas permit a quick evaluation of the potential impact of parameter uncertainty on residual risk without requiring a careful formulation of prior beliefs.

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