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Durations of Nondefault-Free Securities

The Research Foundation of
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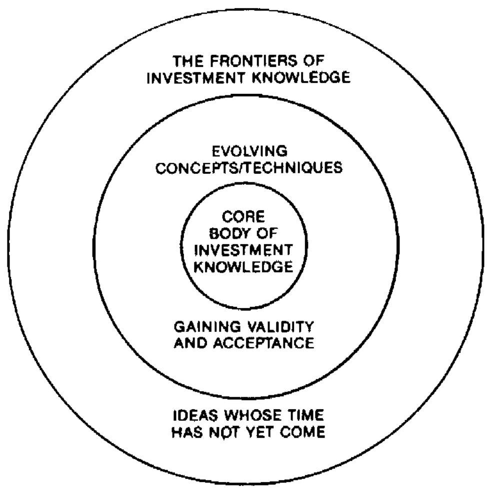

The Research Foundation of
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Foreword

Most duration schemes focus on default- and option-free bonds. But the bond world is now heavily populated with issues where default and its timing are not trivial, and in which options are laced throughout the bond contract, such as in residual mortgages and mortgage-backed securities. The advent of leveraged buyouts and the attendant use of high-yield bonds give rise to an amended concept of default that transcends the usual legal definitions.

A duration measure is needed to analyze debt securities whose promised cash flows may change unexpectedly. Simply put, when cash flows are not nearly as predictable as we might like them to be or when bonds are not free of default, broadly defined, duration measures must account for the *process* of analyzing those now highly unpredictable cash flows between the present and some prospective, but undetermined, time.

Bierwag and Kaufman demonstrate for bond analysts that failure to account for the likelihood of default and its unknown timing, is also to fail in estimating duration accurately. In short, a new risk is introduced into the typical analysis, the undetermined timing of the default prospect. This is a major step in bond portfolio analysis and management.

Charles A. D'Ambrosio, CFA
Director
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1. Introduction

Duration measures the elasticity of the market price of an asset or liability—the present value of an expected stream of cash payments, inflows or outflows—with respect to the market discount rate. At the time a bond is purchased, only the stream of promised future payments is known with certainty. The amount and time pattern of the cash flows actually realized may differ from those promised because of such factors as changes in interest rates, losses from default, and exercises of call, put, or other options. To be useful in designing and implementing bond investment strategies for managing interest-rate risk, measures of duration, whether single or multiple-term, must allow for these uncertainties. Otherwise, *ex ante* duration may not equal *ex post* duration— that is, the investor will experience *stochastic process risk.* This adjustment for uncertainty does not require specifying precisely the unexpected changes in cash flows, but specifying only the stochastic processes governing these unexpected changes.

Most previous constructs of duration have been restricted to default- and option-free bonds (Bierwag et al., *Financial Analysts Journal,* 1983), which are subject to loss only from unexpected changes in default-free interest rates. More recently, because of the sharp jump in prepayments on residential mortgages and thereby also on mortgage-backed securities when interest rates declined sharply in 1985-86, attention has been focused on the problem of computing durations for bonds having call options (Garman 1985; Kopprasch 1987; Toevs 1985). Because prepayments are largely related to changes in interest rates, the unexpected changes in cash flows on these bonds also result solely from unexpected changes in interest rates. Measures of duration have not been developed for debt securities whose promised cash flows may change because of unexpected default by the issuer, although in practice, duration-based strategies are frequently applied to such portfolios, including portfolios of high-yield or "junk" bonds (Feldstein et al. 1983). This study shows that biases are introduced into the computation of duration for nondefault-free bonds when the time pattern of losses from default is not explicitly taken into account. It also examines the problems associated with constructing durations for such bonds, and develops simple, single-factor measures of duration for option-free bonds subject to default risk for some stylized hypothetical stochastic processes governing the time pattern of default losses.

2. The Basics

Bonds subject to default risk trade at higher interest rates than comparable default-free bonds to compensate investors for expected losses resulting from reduced and delayed promised payments. The interest rates on bonds subject to default must be high enough that the expected returns are equal to those on comparable default-free bonds.[1] The difference between the yield to maturity on a bond subject to default and the yield on an otherwise comparable default-free bond is referred to as the ***default yield premium***. The determinants of default yield premiums have been studied extensively in the literature (Altman 1987; Atkinson 1967; Fons, *Journal of Finance*, 1987; Fraine and Mills 1961; Hempel 1971; Hickman 1958; Johnson 1967; Warner, *Journal of Financial Economics*, 1977). Several studies have computed durations for nondefault-free bonds based on the expected loss from default and the

[1]The price (P) of a bond subject to default risk may be expressed as:

$$P = \Sigma S_t (1+r^*)^{-t} = \Sigma (S_t - d_t)(1+r)^{-t}$$

where

S_t = promised cash flows
d_t = expected loss from default
r = risk-adjusted interest rate
r^* = market interest rate, and
t = time to payment.

corresponding default yield premiums (Alexander and Resnick 1985; Chance 1987; Fons, *Working Paper,* 1987). For the same expected losses from default that are impounded in the default yield premium, however, the timing of the reductions in payments may vary greatly, depending on the circumstances of the particular issuer in default. The outcomes of any two bankruptcy proceedings are rarely the same. Thus, although the dollar amount of the default may be expected, the actual timing of the default may be unexpected.

Each time pattern of cash payments translates into a different duration value for a bond for a given stochastic process of default-free interest rates. Although the precise timing of the expected losses is random, it may be described statistically by a stochastic process. Thus, the computation of duration for these bonds must consider the stochastic process governing the timing of default losses for a given expected present value of future payments, as well as the stochastic process governing interest rates.

The importance of including both stochastic processes in the computation of duration may be demonstrated as follows. Assume that the stochastic process driving default-free interest rates is additive and consistent with the one-term Macaulay duration measure so that interest rates on securities of all terms to maturity are the same and the yield curve is flat.[2] Column (b) of Table 1 shows Macaulay durations for par bonds with coupon rates and yields to maturity both equal to 10 percent for progressively longer terms to maturity. Assume also that the expected loss from default on the promised payments is such that the expected

[2]Although more complex and possibly realistic measures of duration have been developed, the single-factor Macaulay measure performs reasonably well in empirical tests (Bierwag 1987; Bierwag, Kaufman, and Latta 1987; Bierwag, Kaufman, Latta, and Roberts 1987).

TABLE 1

Macaulay Durations for Alternative Default and Cash Flow Scenarios for Par Bonds with Yields to Maturity of 10 Percent and Expected Returns of 9 Percent

(a) Maturity	(b) Unadjusted Duration	(c) Maximum Duration	(d) Minimum Duration	(e)	(f)
	D	D_U	D_L	D_U-D	D-D_L
		(years)			
1	0.976	0.976	0.976	0	0
2	1.862	1.878	1.861	0.016	–0.001
3	2.665	2.713	2.661	0.048	–0.004
4	3.393	3.484	3.385	0.091	–0.008
5	4.054	4.196	4.038	0.142	–0.016
6	4.653	4.855	4.627	0.202	–0.026
7	5.197	5.461	5.157	0.264	–0.040
8	5.690	6.018	5.633	0.328	–0.057
9	6.137	6.533	6.060	0.396	–0.077
10	6.543	7.009	6.441	0.466	–0.102
11	6.911	7.449	6.780	0.538	–0.131
12	7.244	7.855	7.082	0.611	–0.162
13	7.547	8.230	7.349	0.683	–0.198
14	7.822	8.576	7.584	0.754	–0.238
15	8.071	8.897	7.790	0.826	–0.281
16	8.296	9.192	7.969	0.896	–0.327
17	8.501	9.465	8.124	0.964	–0.377
18	8.687	9.717	8.256	1.030	–0.431
19	8.856	9.950	8.368	1.094	–0.488
20	9.009	10.165	8.461	1.156	–0.548
21	9.147	10.363	8.537	1.216	–0.610
22	9.273	10.545	8.597	1.272	–0.676
23	9.387	10.713	8.642	1.326	–0.745
24	9.491	10.875	8.675	1.384	–0.816
25	9.584	11.010	8.695	1.426	–0.889

D_U = Duration when default is expected to occur on earliest payments
D_L = Duration when default is expected to occur on last payments.

return is only 9 percent. The resulting default yield premium is 1 percent. The duration of any of these bonds will depend on the pattern of the expected cash payments particular to that bond.

Assume that in the two extremes, the default occurs either (1) on the first coupon(s), but that sufficient payments are made on the later coupons to maintain the risk-adjusted interest rate at 9 percent, or (2) on the last payments in a magnitude necessary to reduce the return from 10 to 9 percent. The Macaulay durations for these assumptions are shown in columns (c) and (d) of Table 1, respectively. (The derivation of these default-time-adjusted durations are shown in Appendix A.) The table illustrates that the durations for the two patterns differ. The early default pattern in column (c) generates consistently longer durations and the later default pattern in column (d) consistently shorter durations. The differences in durations are shown in columns (e) and (f).

There is a hypothetical time pattern of default that is consistent with the initial unadjusted duration values shown in column (b) so that no adjustment need be made in the duration computation. One such pattern reduces the cash payments in each period by an amount that renders the present value of the after-default cash flows on the bond discounted by the risk-adjusted 9 percent interest rate equal to the present value of the initial before-default cash flows discounted by the 10 percent yield to maturity. (This default pattern is derived in Appendix B.) There is no reason to believe that this pattern is appropriate for every bond, however. Thus, the unadjusted duration does not necessarily apply to all bonds subject to default and an investor must specify an expected pattern of defaults for each bond. Of course, the particular pattern specified, just as the expected amount of default, may not be realized. As a result, the investor assumes stochastic process risk for default as well as for interest rates (Bierwag et al., *Journal of Bank Research,* 1983).

Table 2 shows the unadjusted maximum (early default) and minimum (late default) one-factor Macaulay durations for par bonds with 10 percent market yields to maturity when the default yield premium is 2 percent and expected interest return declines to 8 percent. As illustrated in columns (e) and (f), the magnitude of the potential biases from not taking the stochastic process of default explicitly into account is larger than when the default risk premium is only 1 percent (Table 1). It follows that the greater the default yield premium, the greater the need to adjust the bond's duration for the expected timing of default losses.

A more realistic pattern of expected default may be one in which all scheduled coupon and principal payments are delayed K years after default, but are paid in full by the end of the extended maturity. This pattern is shown in Table 3 for 10 percent coupon par bonds with progressively longer terms to maturity having expected after-default returns of 9 percent. Column (c) shows the number of years the payments will be delayed if default occurs immediately to satisfy these conditions. For example, for a bond with 20 years to maturity, the payments will be delayed exactly 1 year; that is, the maturity is extended to 21 years. All default-adjusted durations exceed their corresponding unadjusted durations, and the size of the adjustment increases with term to maturity. The adjustment also exceeds the length of the delay in the payments.

It follows from the above numerical examples that adjusting for stochastic default risk as well as for stochastic term structure risk modifies the computation of the relevant duration. The new measures of duration depend on the stochastic process of default assumed. For example, if the

TABLE 2

Macaulay Durations for Alternative Default and Cash Flow Scenarios for Par Bonds with Yields to Maturity of 10 Percent and Expected Returns of 8 Percent

(a) Maturity	(b) Unadjusted Duration	(c) Maximum Duration	(d) Minimum Duration	(e)	(f)
	D	D_U	D_L	D_U-D	D-D_L
		(years)			
1	0.976	0.976	0.976	0	0
2	1.862	1.896	1.859	0.034	–0.003
3	2.665	2.758	2.657	0.093	–0.008
4	3.393	3.559	3.376	0.162	–0.017
5	4.054	4.313	4.022	0.259	–0.032
6	4.653	5.024	4.600	0.371	–0.053
7	5.197	5.681	5.116	0.484	–0.081
8	5.690	6.299	5.574	0.609	–0.116
9	6.137	6.880	5.979	0.743	–0.158
10	6.543	7.427	6.334	0.884	–0.209
11	6.911	7.936	6.643	1.025	–0.268
12	7.244	8.411	6.911	1.167	–0.333
13	7.547	8.858	7.139	1.313	–0.408
14	7.822	9.278	7.331	1.456	–0.491
15	8.071	9.673	7.490	1.602	–0.581
16	8.296	10.044	7.618	1.748	–0.678
17	8.501	10.389	7.717	1.888	–0.784
18	8.687	10.711	7.790	2.024	–0.897
19	8.856	11.013	7.839	2.157	–1.017
20	9.009	11.297	7.865	2.288	–1.144
21	9.147	11.563	7.871	2.416	–1.276
22	9.273	11.813	7.871	2.540	–1.402
23	9.387	12.047	7.871	2.660	–1.516
24	9.491	12.267	7.871	2.776	–1.620
25	9.584	12.472	7.871	2.888	–1.713

D_U = Duration when default is expected to occur on earliest payments
D_L = Duration when default is expected to occur on last payments.

TABLE 3

Macaulay Durations for Default Patterns On Par Bonds with Yields to Maturity of 10 Percent In Which All Payments Are Delayed by K Years To Generate Expected Returns of 9 Percent

(a) Maturity	(b) Unadjusted Duration D	(c) Delay In Payments K (years)	(d) D_K	(e) D_K-D
1	0.976	0.106	1.082	0.106
2	1.862	0.202	2.065	0.203
3	2.665	0.289	2.959	0.294
4	3.393	0.369	3.773	0.380
5	4.054	0.441	4.516	0.462
6	4.653	0.506	5.193	0.540
7	5.197	0.566	5.813	0.616
8	5.690	0.621	6.380	0.690
9	6.137	0.670	6.899	0.762
10	6.543	0.716	7.374	0.831
11	6.911	0.757	7.810	0.899
12	7.244	0.795	8.209	0.965
13	7.547	0.829	8.576	1.029
14	7.822	0.861	8.913	1.091
15	8.071	0.889	9.222	1.152
16	8.296	0.916	9.507	1.213
17	8.501	0.940	9.768	1.267
18	8.687	0.962	10.008	1.321
19	8.856	0.982	10.230	1.374
20	9.009	1.000	10.433	1.424
21	9.147	1.017	10.620	1.473
22	9.273	1.032	10.793	1.520
23	9.387	1.046	10.951	1.564
24	9.491	1.059	11.097	1.607
25	9.584	1.070	11.232	1.647

D_K = Duration when default is expected to cause all payments to be postponed K years.

assumed process produced defaults only on the last payments, and interest rates were the same for all maturities and change only additively (the Macaulay assumptions,) the single-factor measure of default-adjusted duration (D_A) is given by:

$$D_A = \left[\sum_{1}^{K-1} tS_t\,(1+r)^{-t} + K(1-\alpha)S_K(1+r)^{-K} \right] / P \qquad (1)$$

where

- S_t = the scheduled cash payment at time t,
- K = periods to the first payment that suffers loss from default,
- α = the proportion of the K^{th} payment lost from default,
- P = bond price,
- r = default risk-adjusted yield to maturity, and
- N = term to maturity.

The derivation of this duration measure is shown in Appendix A. In this formulation, partial or total default occurs on the K^{th} payment and total default on all of the remaining N-(K+1) payments in a magnitude that is consistent with the default risk premium included in the initial market yield to maturity. The duration of this equation is shown in Appendix A. In contrast with unadjusted duration, default-adjusted duration requires the specification of both a description of the time pattern of the expected reduced cash flows and the default risk-adjusted discount rate.

If the same stochastic process of interest rates holds but all scheduled payments are delayed for K* years, however, then the correct single-factor measure of duration is:

$$D_A = K^* + \sum_{1}^{N} tS_t(1+r)^{-t} / P \qquad (2)$$

where K* is determined to be consistent with a given default-risk premium included in the market yield to maturity. The derivation of this duration measure appears in Appendix C. Other stochastic processes of default for the same stochastic process of interest rates may result in different single-factor measures of default-adjusted duration. More complex stochastic processes would produce both more complex, multifactor measures of duration and more complex differences in the relevant durations.

3. The Consequences of Ignoring Stochastic Default

The implications for portfolio management of failing to take the stochastic process of default into account in computing durations are easily demonstrated. Assume that the relation between the expected risk-adjusted return on a security or portfolio of securities and duration is linear and may be expressed as (Babcock, "Duration," 1984; Babcock, "Erratum," 1984; Bierwag 1987; Bierwag et al., *Financial Analysts Journal,* 1983):

$$r = r_0 + \frac{(PL - D)}{PL} (r_1 - r_0) \tag{3}$$

where

r	=	expected risk-adjusted return,
r_0	=	initial risk-adjusted yield to maturity,
r_1	=	expected risk-adjusted yield to maturity immediately after purchase,
D	=	Macaulay duration, and
PL	=	investor's planning period.

The bond's interest-rate risk is proportional to the value of $(PL - D)/PL$. To immunize the return on a bond against unexpected changes in risk-adjusted interest rates, an investor must choose a bond with a duration equal to the expected planning period. This reduces the term (PL - D)/P to zero. Assume that the length of the investor's planning period is 8-1/2 years and that the universe of available bonds are those shown in Table 1. If the investor computed simple unadjusted Macaulay durations, he or she would select the 17-year maturity bond, whose duration is 8-1/2 years and equal to the planning period. But if, as discussed above, the stochastic process of default did not reduce every promised payment by the present value of the expected default, the actual value of (PL - D)/PL would not be zero and the investor would not be immunized. Instead, the investor would unknowingly be assuming interest-rate risk.

If all the expected defaults occur on the last payments, the investor wishing to immunize should have selected the 21-year bond, which has an appropriate duration of 8-1/2 years; if all the expected defaults occur on the early payments, the investor should have chosen the 14-year maturity bond, which has a duration of approximately 8-1/2 years for this default pattern. Because any degree of interest-rate risk is measured by the term (PL - D)/PL, similar errors from neglecting the time pattern of default losses occur for investors who prefer to pursue active interest-rate risk strategies rather than to immunize, and to assume specific nonzero risk-adjusted interest-rate risk exposures on bonds subject to default risk.

4. Conclusion

This study demonstrated that for securities subject to default, durations—whether single or multiple-term—computed on the basis of market rates of interest must be adjusted to reflect the expected timing of the expected losses from default. This requires both estimation of the expected loss from default to obtain the risk-adjusted interest rate and a chronological description of the reduced cash flows. Although including the stochastic processes generating both interest rates and the time pattern of default losses complicates the derivation and computation of duration, it is necessary to do so to minimize total stochastic-process risk and to manage intelligently portfolios of nondefault-free bonds. Because relatively little is known about the timing of reductions in payments for bond issues after default, much additional research on the bankruptcy process is required. A primary purpose of this study is to motivate such efforts.

Appendix A

Default-Adjusted Durations for Default Patterns on the Earliest and Latest Payments for a Given Default Risk Premium

If the expected default patterns occur on the early (late) payment dates, then the duration of the entire income stream is larger (smaller). This result is shown to be true for a given default risk premium included in the market yield to maturity. The price of a bond is given as

$$P = \sum_{1}^{N} S_t(1 + r_t^*)^{-t} = \sum_{1}^{N} (S_t - d_t)(1 + r)^{-t} \quad \text{(A1)}$$

where

S_t = the promised cash flow at date t,

d_t = the expected default at date t,

r^* = the market yield to maturity,

r = the risk-adjusted yield to maturity, and

N = the maturity of the bond.

The Macaulay duration is

$$D = \sum_{1}^{N} t\, S_t(1 + r^*)^{-t}/P. \quad \text{(A2)}$$

The duration adjusted for the risk of the expected default is

$$D_A = \sum_{1}^{N} t(S_t - d_t)(1 + r)^{-t}/P. \quad \text{(A3)}$$

To find the maximal adjusted duration, we let

$$d_t = \begin{cases} S_t, & t = 1,2,\ldots,K\text{-}1 \\ \alpha S_K, & t=K \\ 0, & t = K+1,\ldots,N \end{cases} \tag{A4}$$

for some α ($0 \leq \alpha \leq 1$) and K such that (A1) holds for the given values of r*, r, and N. In this way, all of the expected defaults occur on the earliest possible dates consistent with the pricing equations. Substitution of (A4) into (A1) allows us to define

$$F(\alpha,K) = (1-\alpha)S_K(1+r)^{-K} + \sum_{K+1}^{N} S_t(1+r)^{-t}, \; K+1<N. \tag{A5}$$

Here, $F(\alpha,K)$ is the value of the income stream implied by the expected defaults in (A4). If (α,K) is chosen correctly, then $F(\alpha,K) = P$. The function $F(\alpha,K)$ is a decreasing function of K and α. When $K = 1$ and $\alpha = 0$, $F(0,1) > P$, and when $K = N$ and $\alpha = 1$, $F(1,N) = 0$. Therefore, there is some value, (α^*, K^*), at which $F(\alpha^*,K^*) = P$. To find these values, we may proceed iteratively by letting $\alpha = 0$, and increasing K, integer by integer, until we have some K* at which

$$F(0,K^*-1) \leq P \leq F(0,K^*). \tag{A6}$$

Noting that $F(1,K^*-1) = F(0,K^*)$, it follows that there is some α^* at which $F(\alpha^*,K^*-1) = P$. That is,

$$(1-\alpha^*)S_{K^*-1}(1+r)^{-K^*+1} + F(0,K^*) = P, \tag{A7}$$

or

$$(1-\alpha^*) = \frac{P - F(0,K^*)}{S_{K^*-1}(1+r)^{1-K^*}} \tag{A8}$$

Using (α*,K*) to compute the expected defaults in (A4) and substituting these defaults into (A3) gives us the maximal duration consistent with the pricing equation, (A1).

To find the minimal adjusted duration, we let

$$d_t = \begin{cases} 0, & t = 1,2,\ldots, K\text{-}1 \\ \alpha S_K, & t = K, \\ S_t, & t = K+1,\ldots, N \end{cases} \qquad \text{(A9)}$$

for some $\alpha(0 \leq \alpha \leq 1)$ and K such that (A1) holds for the given values of r*, r, and N. In this way, all of the expected defaults occur in the latest possible dates consistent with the pricing equations. Substitution of (A9) into (A1) allows us to define

$$G(\alpha, K) = \sum_{1}^{K-1} S_t(1+r)^{-t} + (1-\alpha)S_K(1+r)^{-K}. \qquad \text{(A10)}$$

Here, G(α,K) is a monotonic increasing function of K and a monotonic decreasing function of α. We can observe that G(0,N) > P > G(1,1) = 0. Therefore, there are values (α*,K*) at which G(α*,K*) = P. Assuming α = 1, we may proceed monotonically until we have found a value of K* such that

$$G(1,K^*) \leq P \leq G(1,K^*+1), \qquad \text{(A11)}$$

so that for some α*, G(α*,K*) = P. That is,

$$(1 - a^*) = \frac{P\text{-}G(1,K^*)}{S_{K^*}(1+r)^{-K}} \qquad \text{(A12)}$$

Using (α*,K*) to compute the expected defaults in (A9) and substituting these defaults into (A3) gives us the minimal duration consistent with pricing equation (A1).

The procedures for finding (α*,K*) in the case of coupon bonds allow for explicit expressions for F(α,K) and G(α,K).

For example, if we (a) allow for semi-annual discounting, (b) allow for semi-annual coupon payments, and (c) measure maturity in six-month periods, then

$$F(\alpha,K) = (1+r/2)^{-(K-1)}p(c,r,N\text{-}K+1)$$

$$-\alpha(c/2)F(1+r/2)^{-K} \tag{A13}$$

where (c/2) is the semi-annual coupon rate, F is the face value of the bond,and p(c,r,N-K+1) is the price of a bond with annual coupon rate c, annual yield to maturity r, and maturity N-K+1. Similarly,

$$G(\alpha,K) = A(c,r,K) - \alpha F(1+r/2)^{-K}(c/2) \tag{A14}$$

where A(c,r,K) is the value of an annuity consisting of (c/2)F dollars every six months for K six month periods.

Appendix B

Default-Adjusted Durations for Default Patterns Consistent with Unadjusted Durations

It is clear from Tables 1 and 2 that there are expected default patterns in which the adjusted duration is identical to the unadjusted duration. Some of these special cases are of interest.

The price of the security in those cases in which the promised payments are subject to default may be expressed as

$$P = \sum_{1}^{N} S_t(1+r^*)^{-t} = \sum_{1}^{N} (S_t-d_t)(1+r)^{-t} \qquad \text{(B1)}$$

where S_t is the promised payment at date t, d_t is the expected default, r^* is the market yield to maturity, r is the default-risk-adjusted yield to maturity, and (r^*-r) is the default risk premium included in the market yield to maturity. The unadjusted duration is

$$D = \sum_{1}^{N} tS_t(1+r^*)^{-t}/P \qquad \text{(B2)}$$

and the adjusted duration is

$$D_A = \sum_{1}^{N} t(S_t-d_t)(1+r)^{-t}/P. \qquad \text{(B3)}$$

An obvious default pattern — $d_1, d_2, \ldots, d_N$ — in which $D = D_A$ is one in which

$$\frac{S_t - d_t}{(1+r)^t} = \frac{S_t}{(1+r^*)^t}, \; t=1,2,\ldots,N. \qquad \text{(B4)}$$

Here, the present values of the payments after the expected defaults are equal to the present values of the payments before the expected defaults. If the income stream, $S_1, S_2, \ldots, S_N$, can be broken into its separate components and be bought and sold separately, then equation (B4) gives two alternative ways for evaluating each of the cash flows in equilibrium for the given expected default pattern.

If (1) the term structures —r^* and r— are flat, (2) the term structures shift only in an additive manner, and (3) cash flow components are separable and sold separately, then no other expected default pattern is consistent with an equilibrium in which equation (B4) holds. On the other hand, if the components of the income stream are not separatable and sold separately, than a variety of expected default patterns are possible because then equation (A4) need not be required to hold in an equilibrium for those expected defaults.

From this it follows that stochastic processes over the default patterns and interest rates and the degree to which income stream components are bought and sold separately are relevant features of an equilibrium model of the term structure and of the corresponding relation between the adjusted and unadjusted durations.

Appendix C

Default-Adjusted Durations for Default Patterns in Which All Scheduled Payments Are Delayed

The value of the promised income stream of a bond may be expressed as

$$P = \sum_{1}^{N} S_t(1+r^*)^{-t}, \tag{C1}$$

where S_t is the promised cash flow at time t, N is the maturity, and r* is the market yield to maturity on the bond. If the expected default pattern is one in which all payments are delayed by K years but on which there is some interest accumulation, the expected cash flows, after default, become

$$\bar{S}_t = 0, \ t = 1,2,\ldots, K \tag{C2}$$

$$\bar{S}_t = (1+\gamma r^*)^K S_{t-K}, \ t = K+1,\ldots,N+K$$

where γr^*, $0 \leq \gamma \leq 1$, is the interest rate on delayed payments. If r is the risk-adjusted discount rate, then

$$P = \sum_{K+1}^{N+K} \bar{S}_t(1+r)^{-t}, \tag{C3}$$

where r*-r is the default risk premium included in the market yield to maturity r*. Substitution of (C2) into (C3) gives

$$P = \sum_{K+1}^{N+K} (1+\gamma r^*)^K(1+r)^{-t} S_{t-K} \tag{C4}$$

The duration of this income stream, after the expected defaults, may be computed as

$$D_K = \sum_{K+1}^{N+K} tS_{t-K}(1+r)^{-t}(1+\gamma r^*)^K/P \qquad \text{(C5)}$$

$$= \sum_{K+1}^{N+1} tS_{t-K}(1+r)^{-t} / \sum_{K+1}^{N+1} S_{t-K}(1+r)^{-t}$$

$$= \sum_{1}^{N} (t+K)S_t(1+r)^{-t} / \sum_{1}^{N} S_t(1+r)^{-t}$$

We can thus write the adjusted duration as

$$D_K = K + \sum_{1}^{N} tS_t(1+r)^{-t} / \sum_{1}^{N} S_t(1+r)^{-t}. \qquad \text{(C6)}$$

The results in Table 3 are derived by assuming $r = 0.0$, $r = 0.09$, and the cash flows are such that the bond sells at par, ($P = 100$). The Macaulay duration of this bond is

$$D = \sum_{1}^{N} tS_t(1+r^*)^{-t} / \sum_{1}^{N} S_t(1+r^*)^{-t}. \qquad \text{(C7)}$$

Given that duration is a decreasing function of the rate of interest and that $r^* > r$, it follows that

$$D < D_K \qquad \text{(C8)}$$

or that

$$D_K > D + K, \qquad \text{(C9)}$$

which is clearly shown to be the case in comparing columns (e) and (c) of Table 3. In words, the default-adjusted duration increases by more than the delay in payment.

It is also apparent that the expected default-adjusted duration is not affected by γ or the extent to which interest is paid on delayed payments.

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