THE MULTI-INDEX MODEL AND PRACTICAL PORTFOLIO ANALYSIS

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FOREWORD

This publication represents the fourth in the Occasional Paper Series, a series that is intended to cover a variety of topics of interest to financial analysts, presented in a length that is longer than a journal article but shorter than a full length book.

Major new approaches and techniques to investment management have been introduced in recent years to assist portfolio managers in estimating returns and in controlling risk levels in their portfolios. The linear single-index model that uses a beta coefficient has received considerable attention and discussion among professional analysts. The author of this paper reviews three major portfolio models and presents a multi-index model that has a potential of being a significantly useful extension to the existing single-index model and to beta theory.

We are extremely grateful to the author, Dr. James L. Farrell, Jr., C.F.A., for his outstanding piece of research and his well written paper.

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W. Scott Bauman, C.F.A.
Executive Director

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July, 1976 *J. L. F.*

I. Introduction

Modern portfolio analysis is concerned with grouping individual investments into an efficient set of portfolios. A portfolio is defined as efficient if (and only if) it offers a higher overall expected return than any other portfolio with comparable risk. Three analytical methods for developing efficient sets of portfolios are: (1) Markowitz model;[1] (2) Sharpe single-index model;[2] and (3) Cohen and Pogue's multi-index model.[3]

The Markowitz model established the basic framework for modern portfolio analysis and provides the most accurately developed set of efficient portfolios. The size and complexity of the model, however, makes it virtually inapplicable for practical use. The Sharpe model economizes on inputs and computer time but neglects important relationships among securities. Failure to assess these relationships results in a set of portfolios that is less than truly efficient. Cohen and Pogue's multi-index model provides a means of accounting for these relationships while at the same time achieving substantial input and computational savings over the Markowitz technique. The multi-index model should thus be the preferred technique for practical portfolio analysis.

The essential problem in using the multi-index model is developing appropriate inputs. Industry groups have been tested in the model but do not provide inputs that allow it to operate at maximum efficiency. Needed are indexes of stock returns that are homogeneous in the sense of being significantly correlated within their own grouping and, at the same time, generally independent of other groups.

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1. Harry M. Markowitz, *Portfolio Selection: Efficient Diversification of Investments*, (New York: John Wiley & Sons, Inc.), 1959.

2. William F. Sharpe, "A Simplified Model for Portfolio Selection," *Management Science*, January 1963, pp. 277-293.

3. Kalman J. Cohen and Jerry A. Pogue, "An Empirical Evaluation of Alternative Portfolio Selection Models," *Journal of Business*, April 1967, pp. 166-193.

The writer has shown that it is in fact possible to develop stock groupings with such desirable collinearity characteristics.[4] A study was undertaken to use indexes constructed from these stock groupings as inputs to the multi-index model, and to test that model's performance against the performance of the Sharpe and Markowitz models.

This study proceeds as follows. Section II describes the three portfolio selection models and compares them with regard to inputs required and accuracy of representing the relationships among securities. Section III discusses the results of a prior test of the effectiveness of the multi-index model in generating efficient portfolios and also describes the need for developing appropriate inputs to maximize the effectiveness of the model. Section IV discusses techniques for developing appropriate inputs and describes the sort of indexes that were developed by analyzing a sample of representative companies. Section V uses these inputs to test the performance of the multi-index model against the single-index and Markowitz models. Section VI provides a summary and conclusions.

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4. James L. Farrell, Jr., "Analyzing Covariation of Return to Develop Homogeneous Stock Groupings," *Journal of Business*, April 1974.

II. Portfolio Selection Models

As noted in the introduction, the three analytical methods for developing efficient sets of portfolios are: (1) Markowitz model; (2) Sharpe single-index model; and (3) Cohen and Pogue's multi-index model.[1] This section describes each model as to the method of generating efficient portfolios as well as inputs required for the analysis. It also compares the three types of models both as to ease of practical implementation and as to facility in representing the interrelationships among securities.

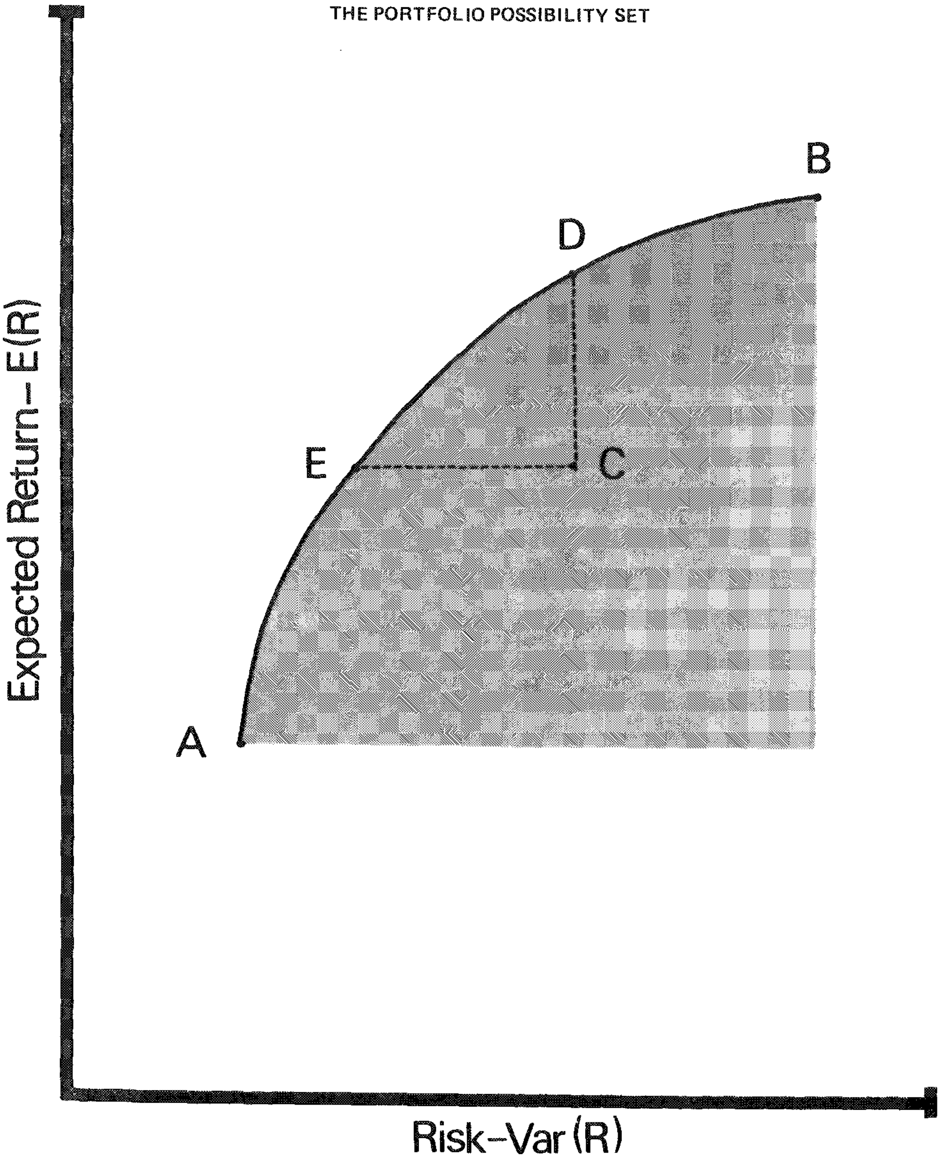
Markowitz Model

Markowitz pioneered in developing a well defined theoretical structure for portfolio analysis that can be summarized as follows. First, the two relevant characteristics of a portfolio are its expected return and some measure of the dispersion of possible returns around the expected return, the variance being analytically the most tractable. Second, rational investors will choose to hold efficient portfolios, which are those that maximize expected returns for a given degree of risk or, alternatively and equivalently, minimize risk for a given expected return. Third, it is theoretically possible to identify efficient portfolios by the proper analysis of information for each security on expected return, the variance in that return, and the co-variance of return for each security and that for every other security. Finally, a specified, manageable computer program can utilize inputs from security analysts in the form of the three kinds of necessary information about each security in order to specify a set of efficient portfolios. The program indicates the proportion of an investor's fund that should be allocated to each security in order to achieve efficiency, i.e., the maximization of return for a given degree of risk or the minimization of risk for a given expected return.

In describing the Markowitz model, it is useful to illustrate first the concept of efficiency by means of Figure 1. The vertical axis refers to expected return while the horizontal axis refers to risk as measured by the variance of return (the standard deviation of return, which is

1. Cohen and Pogue actually developed two forms of the multi-index model—the covariance form and the diagonal form. Both forms are quite similar and to simplify the exposition only the covariance form is analyzed.

FIGURE 1
THE PORTFOLIO POSSIBILITY SET



the square root of the variance, is used alternatively as a measure of risk). The shaded area represents the set of all the possible portfolios that could be obtained from a given group of securities. Associated with each possible portfolio will be a certain level of return and a certain risk. Thus, each portfolio is represented by a single point in the shaded area of Figure 1.

Note that the efficient set is represented by the upper left-hand boundary of this shaded area between points A and B. Portfolios along this efficient frontier dominate those below the line. Specifically, these efficient portfolios offer higher return than those at an equivalent level of risk or alternatively entail less risk at an equivalent level of return. For example, note that portfolio C, which does not lie on the efficient boundary, is dominated by portfolios D and E, which do lie on the efficient boundary. Portfolio D offers greater return than C at the same level of risk while portfolio E entails less risk at the same level of return than portfolio C.[2]

As noted, an efficient portfolio (or any portfolio for that matter) is described by the list of individual securities contained in the portfolio as well as the weighting that each security comprises in the portfolio. The estimated or expected return of the portfolio is, in turn, merely a weighted average of the expected returns of the individual securities comprising the portfolio. Calculation of the expected return can be most easily illustrated for the case of a two-security portfolio. Using (X_i) to represent the security's proportion of the portfolio and (R_i) the expected return, the expected return of the portfolio (R_p) is calculated as follows:

$$R_p = X_1R_1 + X_2R_2 = \sum_{i=1}^{2} X_iR_i$$

While the expected return of a portfolio can be obtained directly, the variance (or risk) of a portfolio is not simply a weighted average of the variances of the individual securities in the portfolio. There is also a need to consider the relationship between each security in the portfolio and every other security as measured by the covariance of return. The method of calculating the variance of a portfolio can again most easily be illustrated for the case of a two-security portfolio. Using Var (R_i) to represent the variance of each security, Cov (R_1R_2) to represent the covariance between the two securities, and again using X_i to represent the proportion that each security represents in the portfolio, the calculation of the portfolio variance (Var R_p) is as follows:

2. The particular portfolio that an individual investor selects from the efficient frontier depends on that investor's degree of aversion to risk. In more technical terms, it depends on the nature and shape of the investor's risk-return utility function.

$$Var(R_p) = X_1^2 Var(R_1) + X_2^2 Var(R_2) + 2X_1 X_2 Cov(R_1 R_2)$$

In words, the variance of the portfolio is the weighted sum of the variances of the individual securities plus twice the covariance between the two securities.

To estimate the expected return and variance of a two-stock portfolio, five estimates are needed: expected return for each stock; variance of return for each stock; and covariance of return between the two stocks. Generalizing to the case of N stocks, there would be need for N return estimates, N estimates of the variance, but a total of

$$\frac{N(N-1)}{2}$$

covariance estimates. For example, analyzing a set of 100 stocks would require 100 return estimates, 100 variance estimates, and 4,950 covariance estimates for a total of 5,150 estimates. Note that the task of estimation is increased considerably by the need to consider explicitly the interrelationship among securities as represented by the covariance. This, as might be suspected, creates one of the main problems of practical implementation.

Given estimates of returns, variances, and covariances for the securities in the universe under consideration, the efficient set of portfolios is generated by means of a programming routine known as quadratic programming. A detailed description of the routine is not essential to the purpose of this study and will only be summarized as follows. Essentially, the program is constructed to minimize the risk of a portfolio at a given level of return, i.e., develop the efficient portfolio at a given return level (say for example, at 5 percent, or 10 percent or 20 percent). The program develops minimum risk portfolios at different levels of return, in each case specifying the securities and their weightings in the portfolio at that level of return. Proceeding in this fashion, the program develops a series of portfolios differing in risk and return that trace out an efficient frontier similar to the one illustrated by the curve AB in Figure 1.

While the Markowitz model provides the most complete procedure for developing efficient portfolios, substantial problems of practical implementation emanate mainly from the overwhelming burden of developing input estimates for the model. As noted, analysis of a universe of 100 stocks would require 5,150 different estimates. The task of collecting the estimates for these 5,150 statistics is further complicated by the fact that few individuals are capable of estimating such sophisticated measures as variances and covariances. In addition, the quantity of data required taxes the memory capacity of even the largest computers.

Furthermore, the co-ordination of this data-gathering procedure presents difficulties. Most securities research departments are organized so that specialists are assigned to the coverage of an individual industry or small group of industries. In turn, this specialization means that individual analytical personnel generally have little knowledge of the characteristics of industries other than their assigned industry. Thus obtaining estimates of relationships across industries is difficult. For example, the electronics specialist is likely to find it difficult to assess the degree of comovement between his assigned industry and others such as the food or chemical industries.

Single-Index Model

The Sharpe single-index modification of the Markowitz model circumvents the difficulty of dealing with a great number of covariances by providing a simplified method of representing the relationships among securities. The fundamental concept underlying Sharpe's simplified approach to portfolio analysis is that the only form of comovement between securities comes from a common response to a general market index such as the Standard & Poor's (S&P) 425. More specifically, it is assumed that the return (R_i) on any security is determined by random factors and a linear relationship with the market index (R_m) of the following form:

$$R_i = A_i + B_i R_m + C_i$$

Note that this model is simply a regression equation with B_i representing the slope coefficient and A_i representing the intercept from the regression. In terms of the single-index model, the slope measures the responsiveness of the security's return to movements in the market, while the intercept measures the component of security return that is independent of the market return. The C_i or residual term is assumed to equal zero on average, as is standard in regression analysis. Most importantly, it is assumed that the residuals are uncorrelated across securities, i.e., E (C_iC_j) = 0. This is in keeping with the single-index model assumption that the sole source of comovement among securities is due to the general market; so that once this influence has been removed, the expectation is for no correlation among the residuals of different securities.

Given the assumption that the residuals are uncorrelated (cross-sectionally independent), the covariance between one security and any other security in the universe (Cov R_iR_j) can be derived from the basic inputs to the model. Specifically, the only need is for an estimate of each security's market responsiveness (B_i) and an estimate of the market variance (Var R_m). The covariance between the two securities can then be calculated according to the following formulation:

$$\text{Cov } R_iR_j = B_iB_j \text{ Var } R_m$$

In principle, it would be possible to proceed in this fashion and calculate the covariance for each security in the universe. However, Sharpe has noted a simplification that circumvents the need for all these calculations and yet provides an equivalent representation of the covariance among securities. Again, detailed discussion of this simplification is not essential to this study.

As a result of eliminating the need to consider the comovement among securities (except that due to general market movements), the data requirements for a portfolio analysis using the Sharpe model are substantially less than those for the Markowitz model. Only three estimates are required for each security to be analyzed: specific return (A_i); measure of responsiveness to market movements (B_i); and the variance of the residual return (Var C_i). For example, analysis of a universe of 100 securities would require only 300 estimates (three estimates x 100 stocks). This reduction in inputs for individual securities is at the cost of two additional estimates: an estimate of the return (R_m) and variance (Var R_m) of the market index. The net requirement of 302 estimates for the analysis is less than one-tenth the 5,150 required for the full Markowitz formulation. Furthermore, the Sharpe model has the important advantage that analysts need to provide information only for the securities they follow.

While Sharpe's single-index model provides a modification of the Markowitz model that economizes on the number of inputs as well as computer time required to perform a portfolio analysis, the formulation is an oversimplification. The Sharpe model assumes that the only effects common to all securities arise solely from general market movements, but there is evidence of other less significant but still substantial factors affecting the returns of all securities. (In the language of the single-index model, there is a significant violation of the specification of uncorrelated residuals). More specifically, King[3] has documented the existence of industry effects and, that other broader-than-industry effects which influence the returns on individual securities. Failure of the single-index model to incorporate these important relationships is thus likely to degrade portfolio efficiency.

3. Benjamin F. King, "Market and Industry Factors in Stock Price Behavior", *Journal of Business*, January 1966, pp. 139-190.

Multi-Index Model

Cohen and Pogue's multi-index model provides a means of incorporating these effects into a portfolio analysis, while at the same time achieving some substantial savings over the general Markowitz approach. The multi-index model is similar to the single-index model in that the return of each security under consideration is assumed to be a linear function of some index. The multi-index model, however, assumes that the universe of securities is composed of components from several classes or industries and relates each security to its respective class or industry rather than to one general market index. In particular, the return on each security, (R_i), is related to its class or industry, (R_k), in the following manner:

$$R_i = A_i + B_i R_k + C_i$$

As in the case of the single-index model, this is simply a regression equation with B_i representing the slope coefficient and A_i representing the intercept from the regression. The slope measures the responsiveness of the security's return to the return on the class or industry index, while the intercept measures the component of security return that is independent of class or industry return. The C_i or residual term is assumed to equal zero on average, as well as being specified to be uncorrelated across securities in the class or industry, i.e., the residual terms of the two stocks (stocks i and j) in the same industry are uncorrelated, $E\ (C_i C_j) = O$.

Note that the covariance between securities within a class or industry can be derived in essentially the same fashion as for the single-index model. As a result, the comovement of companies within the same class or industry depends solely on the response of the companies to their class or industry index much as in the case of the single-index model specification.

On the other hand, the comovement of securities from different classes depends on each company's response to its class or industry index and the extent to which the classes or industries move together. The multi-index model allows explicitly for this comovement by considering the covariance between industry or class indexes in much the same manner as between securities in the Markowitz formulation.[4]

4. The covariance form and diagonal form of the multi-index model differ in the way each specifies interaction among component industry or class indexes. The diagonal form of the multi-index model assumes that the industry or class indexes are related to one another only to the extent that the industries or classes move with the market. The covariance form of the multi-index model captures the full pattern of interaction between industry or class indexes, rather than assuming they are only related through common movement with the market index.

Despite providing for the full covariability among class or industry indexes, the multi-index model still allows for a substantial reduction in inputs. For example, an analysis of 100 securities that could be subdivided into four classes would require the following inputs. The first need is to estimate the specific return (A_i), measure of responsiveness (B_i), and the variance of residual return (Var C_i) for each security with its respective index, or a total of 300 estimates for the universe. The second need is to estimate the return (R_k) and variance (Var R_k) for each of the four class indexes for an additional eight estimates. The final need is to develop explicitly covariances among the four indexes or a total of six different covariances.

The net requirement for an analysis of the 100 securities is 314 estimates, or only slightly larger than the 302 required by the single-index model. It is of course substantially less than the 5,150 required by the Markowitz model for the analysis of the same size universe. At the same time, it would be expected that the multi-index model could incorporate into the analysis those systematic effects among securities neglected by the single-index model and provide almost as exact an analysis as the Markowitz model. It thus seems that the multi-index model would be the preferred model for practical portfolio analysis.

III. Empirical Test and Multi-Index Model Inputs

As noted in the introductory remarks, the major problem in implementing the multi-index model has been in developing appropriate inputs. Cohen and Pogue used industry indexes as inputs to the model and tested its performance against that of the Markowitz model and single-index model. The multi-index model failed, however, to outperform the single-index model because of some basic deficiences of the industry indexes as inputs to the model.

This section describes the Cohen and Pogue empirical test in greater detail and also explains the problems of using industry indexes as inputs to the multi-index model.

This section also reviews the work of Benjamin King on factors explaining the returns of common stocks. This study was important because it provided empirical support for the use of index models for purposes of portfolio selection. The study confirmed that the general market factor is highly important in explaining the returns of common stocks and thus provided support for the use of the single-index model. Correspondingly, the study showed that industry factors can be important in explaining common stock returns and provided support for the use by Cohen and Pogue of industry indexes as inputs to the multi-index model.

While the King study showed the importance of these two factors (market and industry) in explaining common stock returns, indications from the study itself show the need for an additional broader-than-industry factor to explain more completely common stock returns. The remainder of this section discusses some potential bases for introducing an additional factor for explaining stock returns. The section concludes by noting that grouping stocks according to growth, cyclical, and stable characteristics provides the most promising basis for developing a broader-than-industry factor for explaining stock returns.

Empirical Evaluation of Portfolio Selection Models

Cohen and Pogue[1] tested the Markowitz, single-index, and multi-index models to determine the relative efficiency of each in selecting optimum portfolios on both an ex-ante and ex-post basis.

1. Cohen and Pogue, *op. cit.*

The authors utilized historical returns, variances, and covariances among individual stocks as inputs to the Markowitz model. Historical market returns and variances of return as well as historical relationships of individual stocks with the market (i.e., historical A_i, B_i, and Var C_i values) were used as inputs to the single-index model. Cohen and Pogue utilized ten industries classified by traditional industry classifications (two-digit S.I.C. codes) as indexes for the multi-index model. Again, historical industry returns and variances of industry returns as well as historical relationships of stocks with the industry indexes (i.e., historical A_i, B_i, and Var C_i values) were used as data inputs to the multi-index models.

The test indicated essentially that the multi-index model did not provide the expected improved performance over the single-index model on either an ex-ante or ex-post basis. Specifically, Cohen and Pogue found in their evaluation of these models that the ex-ante efficient frontiers generated by portfolios using the single-index model dominated those generated by the multi-index model. In addition, the authors found that the ex-post performance of the index models was not dominated by the Markowitz formulation, and that the performance of the multi-index model was not superior to that of the single-index formulation.

Cohen and Pogue attributed this difference in performance of the models to the relative ability of each to reproduce the true covariance between individual security returns. The authors obtained a measure of this relative ability by comparing the correlation matrix implied by the index models with the true correlation matrix used in the Markowitz model. They found that while the multi-index model most closely represented the true correlations among securities within the same industries, the relationships among securities in different industries were somewhat better represented by the single-index model. Because of the much larger number of inter-industry as opposed to intra-industry comparisons among companies in the sample, the single-index model was found, on the average, to represent better the true correlation matrix.

While Cohen and Pogue's study showed that the single-index model dominated the multi-index model using traditional industry categories as inputs, these results do not necessarily lead to the conclusion that the single-index model is preferable to the multi-index model. The main implication is that industry indexes are basically deficient as inputs to the multi-index model. Specifically the indexes used in the study showed high inter-index correlation even after removal of the general market effect from each index.[2] Use of such highly collinear inputs does not allow the multi-index model to operate at maximum efficiency.

Indexes of stocks are needed that are homogeneous in the sense that they are significantly correlated within their own grouping and, at the same time, independent of other groups.[3] Use of such non-correlated indexes in the multi-index model should lead to a more accurately represented covariance matrix and thereby a set of efficient portfolios superior to the single-index model. As noted earlier, it is a major purpose of this study to develop stock groupings that are homogeneous with respect to collinearity characteristics.

Market and Industry Factors

Before discussing the basis for developing such groupings, it would be useful to review the empirical work of Benjamin King concerning market and industry factors explaining the price behavior of common stocks.[4] King's study hypothesized that the change in the log price of a common stock is the weighted sum of a market, an industry, and a company effect. This hypothesis was based on the observation that investors commonly refer to market-wide or industry-wide price movements of securities; and that companies are typically classified according to industries. King further noted that a widely used method of industry classification was that of the Securities and Exchange Commission and correspondingly desired to test whether a potential industry effect would correspond to the two-digit S.I.C. classifications.

In order to test these hypotheses, King selected a sample of 63 common stocks classified into the following six two-digit S.I.C. industries: (1) Tobacco products; (2) Petroleum products; (3) Metals (ferrous and non-ferrous), (4) Railroads; (5) Utilities; and (6) Retail stores. King then utilized factor analysis to examine the covariation of these 63 common stocks over the 1927-60 period. This analysis indicated that general market effects accounted for roughly 50 percent of the variance in a security's log price over the full period of the study. Industry effects accounted for approximately 10 percent of the variance in log price. The 40 percent remainder of the variance was ascribed to effects unique to an individual security. The stock group-

2. This correlation is noted in table 3 on page 177 of the Cohen and Pogue study showing the frequency distribution of the correlation coefficients of residuals for the ten industry indexes. This table indicates that 40 percent of the residuals showed positive correlation with approximately 15 percent of the residual pairings exhibiting correlation coefficients of 0.50 or larger.

3. Independent in the sense that the residuals of these indexes (after removal of the general market effect) display no correlation.

4. King, *op. cit.*

ings that emerged from the analysis of stock price behavior corresponded to the hypothesized two-digit S.I.C. industry classifications.

While King's results indicate that market and industry effects are important factors explaining the change in stock prices, indications are that the three effects—market, industry, and company—may not be a sufficient number to account for the complex interrelationship of security price changes. First, an analysis of four subperiods by King indicated a successive decline in the importance of the market factor over the total period of the study, as may be noted by the following subperiod statistics: (1) 58 percent, 1927-35; (2) 56 percent, 1935-1944; (3) 41 percent, 1944-52; (4) 31 percent, 1952-60. The decline in the importance of the market factor was also detected in Blume's study.[5] Blume, using regression analysis, found that a market index accounted for an average of 47 percent of the variability of monthly returns on 251 securities over the 1927-60 period, but that the proportion of explained variance declined in each of successive quarterns from 53 percent, to 50 percent, to 38 percent, and finally to 26 percent. This non-stationarity over time of the market effect led King to suggest that other factors may be important in explaining the price of common stocks as reflected by the following statement: "Hence the apparent diminution of the influence of the market implies an increase in relative importance of the unique parts of the variance, or at least those parts that are explained by factors in addition to market and industry."[6]

In addition, the cluster analysis employed by King indicated that some industry groupings showed sufficient correlation to warrant possible consideration as single rather than separate clusters.[7] This correlation among industry groups could be noted from the fact that the cluster procedure continued for several passes after the six hypothesized industries had emerged on the 56th pass of the routine. For example, King's results showed a clustering of the tobacco and retail industries at a positive correlation between 0.15—0.20 and a further clustering of this group with the utility industry at a positive correlation between 0.10—0.15. In addition, the rails and metals industries clustered together with a positive correlation coefficient between 0.10—0.15. The cluster routine then terminated at the 59th pass (no further positive correlations) to provide three separate groups: (1) oil industry; (2) rail and metal industries cluster; and (3) tobacco, retail, and utility industries cluster.

5. Marshall Blume, "Assessment of Portfolio Performance: An Application of Portfolio Theory," (unpublished dissertation), March 1968.

6. King, *op. cit.*, p. 157.

7. King, *op. cit.*, p. 153.

Finally, an analysis of the correlations among industry factors for the overall 1927-60 period as well as four subperiods showed some significantly positive correlation among separate industry factors.[8] Specifically, the tobaccos and stores factors showed positive correlation in two out of four subperiods and for the overall period showed a correlation of +0.25 which is highly significant statistically for a sample with 403 observations. With the exception of the first subperiod, the tobacco and utility industry factors showed positive correlation that was generally statistically significant. Utilities and stores showed positive correlation in two out of four subperiods. Finally, metals and rails showed positive correlation in three out of four subperiods.

As in the case of the cluster analysis, the positive correlation among the utilities, stores, and tobacco factors, as well as the rails and metals factors, indicates the possibility that two separate groups might be formed from these five industries. In addition, the predominantly negative correlation among these groups, as well as with the oil industry, indicates that it might be possible to form a total of three separate groups from the six industries analyzed by King. These results, in conjunction with the cluster analysis results, again suggest the possibility that another factor—broader than the industry factor and in addition to the market and company factors—is needed to explain the variation in log prices of common stocks.

Potential Additional Factor

While the preceding analysis of King's factor analysis results is suggestive of the need for a factor in addition to those of market, industry, and company, the most appropriate basis on which to introduce an additional factor is not immediately apparent. King's method of amalgamating companies to form industry factors suggests amalgamating industry factors to form group factors. Accordingly, we first consider the potential for classifying stocks according to several more conventional economic methods and then describe a method that seems to be the most promising for developing an additional broader-than-industry factor for grouping stocks.

Several more conventional methods for classifying companies or industries by economic characteristics are: (1) dependence on a certain category of spending such as consumer spending, business (capital) spending, or government spending; (2) product similarity such as durable or nondurable goods; or (3) stage in the manufacturing cycle such as raw material, intermediate, or final product. Due to these economic similarities, it might be expected that the earnings patterns

8. King, *op. cit.*, p. 156.

of companies within the same economic categories would behave similarly. The extent to which the prices of stocks coincided with the company's earning pattern would then determine the degree to which the covariation of stocks could be attributed to economic characteristics of companies.

A paper by Granger and Morgenstern[9] is relevant to the subject of the extent to which price patterns of stocks adhere to such methods of classification. Granger and Morgenstern used cross-spectral methods to analyze price indexes and, to a limited extent, to analyze the price behavior of individual stocks. The most pertinent result for purposes of this study is the report on the coherences among changes in the several component indexes (mainly series categorized by economic characteristic) of the S.E.C. Weekly Composite Index (1939-61). The coherence is defined as an index of association between components at the same frequency for two series.

The results of the authors' analysis were consistent with expectations in the cases where price series such as (1) manufacturing and durable goods, (2) durable goods and motor vehicles, (3) transportation and rails showed a high degree of covariation, while series such as (1) manufacturing and utilities, (2) utilities and mining, and (3) mining and trade, finance and services showed an expected low degree of covariation. However, the low degree of covariation among the following series was contrary to expectation: (1) durable goods and radio, television, and communications equipment; (2) motor vehicles and radio, television, and communications equipment; (3) transporation and air transportation. The evidence concerning the covariation of price series according to these economic classifications was thus mixed, as there were almost as many instances of price series moving in an unexpected direction as there were series showing the expected degree of covariation.

A paper by Feeney and Hester[10] is also relevant to the classification of security price covariation according to economic characteristics. Feeney and Hester utilized principal components analysis to study the 30 component stocks of the Dow Jones Industrial Average for non-overlapping quarters during the period 1951-63. The authors extracted the first two principal components from the following three matrices: (1) covariance matrix; (2) correlation matrix; and (3) correlation matrix with the time trend in the original data removed.

An examination (by the authors) of the signs and weights of the components derived from these three matrices indicated that the

9. C.W.J. Granger and O. Morgenstern, "Spectral Analysis of New York Stock Market Prices," *Kyklos*, 1963, pp. 1-27.

10. G.J. Feeney and Donald Hester, "Stock Market Indices: A Principal Components Analysis," (unpublished preliminary paper), 1964.

following DJI stocks showed a relatively high degree of similarity: AT&T, Eastman Kodak, General Foods, Procter and Gamble, Sears, and Woolworth. Feeney and Hester noted that these stocks were perhaps the most consumer oriented of the Dow Jones stocks and conjectured that the extracted components discriminated between producer and consumer goods industrial stocks. The authors speculated that an explanation for such a phenomenon might be that profits of producer goods firms and consumer goods firms reach peaks at different points in a business cycle. A simple accelerator model might yield such a result.[11]

The existence of a potential accelerator effect in the data was, however, only partially substantiated as the producer oriented stocks such as Anaconda, Bethlehem Steel, International Paper, and Swift (Esmark) did not display a high degree of similarity across extracted components. Also, in order for such an accelerator effect to hold true, it must be assumed that stock prices are closely related to current earnings; the use of reported earnings rather than expected earnings in empirical tests of cost of capital models has been a source of criticism. Finally, Feeney and Hester's conjecture of an accelerator effect was based entirely on a preliminary examination of the data and no effort was made (on the part of the authors) to carry this analysis further. As in the case of the Granger and Morgenstern analysis, the existence of a grouping of stocks by these more conventional economic classifications is not strongly supported.

While tests of these more conventional methods of economic grouping have not been fruitful, one that is untested and yet seems promising is grouping stocks according to growth, cyclical and stable characteristics. Investors generally consider growth stocks to be represented by companies expected to show an above average rate of secular expansion. Cyclical stocks are defined as those of companies that have an above average exposure to the vagaries of the economic environment. Earnings of these companies are expected to decline more than average in a recession and to increase more than average during an expansion phase of the business cycle. Stable stocks are those of companies whose earning power is less affected than the average company by the economic cycle. Earnings of these companies are expected to show a below average decline in a recession and a less than average increase during the expansion phase of the business cycle.

11. Feeney and Hester, *op. cit.*, p. 22.

IV. Empirical Test of Stock Grouping Hypothesis

This section tests whether the price action of stocks conforms to classes of growth, cyclical, and stable stocks.[1] The criterion for judging whether stock price movements conform to this method (or any other for that matter) is a high degree of association between the price movements of stocks within a class and a low degree of association with other classes of stocks. More specifically, growth stocks would be expected to be highly related to other growth stocks, cyclicals with other cyclicals, and stables with stables, but growth stocks should be unrelated to cyclicals and stables, and these in turn should be unrelated to each other. Should groups of stocks meet this criterion, it would be appropriate to add a broader-than-industry factor to the market and industry factors previously determined by King.

Selection of Test Sample

To test whether stocks group according to this broader-than-industry method of classification, a sample of 100 common stocks was selected for analysis (Table 1). These 100 companies represent 60 separate Standard & Poor's industry classes and 25 of the two-digit S.I.C. categories. All are listed on a national exchange, and 90 of the companies are included in the Standard & Poor's 500 Index. The total capitalization of this sample of 100 companies represents approximately 50 percent of the market value of the S&P 500. Finally, all industrial companies in the sample are included in Fortune's 500 largest industrial enterprises, with the smallest company listed 300th in sales and 400th in assets.

The majority of stocks in the sample of 100 fell readily into one of the three hypothesized stock categories. For example, technologically-oriented companies, such as those in the electronics and office equipment areas, were easily classified as growth stocks. Machinery companies and other heavy equipment manufacturing categories were clearly cyclical stocks, while consumer-oriented companies such as foods and utilities displayed stable characteristics. Categories

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1. This section essentially reviews material from James L. Farrell, Jr., "Homogeneous Stock Groupings: Implications for Portfolio Management," *Financial Analysts Journal*, May-June 1975. Those familiar with this article may only wish to briefly review this section and proceed to the next section.

Table 1
Sample of Stocks
By S.I.C. and Standard & Poor's Industry Classifications

Stock		S&P Industry Classification	S.I.C. Code
1.	Hewlett-Packard	Electronics	3840
2.	Perkin-Elmer	Electronics	3840
3.	AMP	Electronics	3870
4.	Maryland Cup	Container-Paper	2650
5.	Burroughs	Office Equipment	3570
6.	Ampex	Electronics	3670
7.	Trane	Air-Conditioning	3615
8.	International Telephone	Electrical-Major Co.	3650
9.	Minnesota Mining	Miscellaneous	3990
10.	Baxter Labs	Hospital Supplies	3830
11.	Zenith	T.V. Mfg.	3630
12.	Motorola	T.V. Mfg.	3630
13.	Polaroid	Miscellaneous	3810
14.	Texas Instruments	Electrical-Major Co.	3880
15.	Becton-Dickinson	Hospital Supplies	3830
16.	National Cash Register	Office Equipment	3510
17.	Corning Glass	Miscellaneous	3250
18.	Int'l Flavors & Fragrances	Miscellaneous	2015
19.	International Business Machines	Office Equipment	3570
20.	Avon	Cosmetics	2880
21.	Xerox	Office Equipment	3590
22.	Eastman-Kodak	Miscellaneous	3810
23.	Harcourt, Brace	Publishing	2710
24.	Pan American	Air Transport	4500
25.	UAL, Inc.	Air Transport	4500
26.	United Aircraft	Aerospace	3720
27.	Chesebrough-Ponds	Cosmetics	2880
28.	Nalco	Specialty Chemical	2830
29.	TRW	Miscellaneous	3750
30.	Honeywell	Miscellaneous	3820
31.	Merck	Ethical Drug	2830
32.	Virginia Electric	Electric Utility	5810
33.	American Electric Power	Electric Utility	5810
34.	Central & Southwest	Electric Utility	5810
35.	Florida Power	Electric Utility	5810
36.	Columbia Gas	Natural Gas Dist.	5830
37.	Procter and Gamble	Soap	2840
38.	General Foods	Packaged Food	2050
39.	Chase Manhattan Bank	Bank-New York City	5580
40.	Coca-Cola	Beverage-Soft Drink	2086
41.	Transamerica	Insurance-Life	5540
42.	Household Finance	Finance-Consumer Loan	5520
43.	C.I.T.	Finance-Commercial Loan	5510
44.	Northwest Bank Corp.	Bank-Outside N.Y. City	5580
45.	CPC International	Corn Refiner	3430
46.	Gillette	Miscellaneous	3430
47.	Quaker Oats	Packaged Food	2050
48.	Campbell	Canned Food	2030
49.	Kellogg	Packaged Food	2050
50.	Hershey	Confectionary	2070
51.	Reynolds	Tobacco	2110
52.	American Home Products	Proprietary Drug	2830
53.	Kraftco	Dairy Product	2060
54.	Sears	Retail-Mail Order	5310

(continued)

Table 1 (continued)

Stock		S&P Industry Classification	S.I.C. Code
55.	Federated Department Stores	Retail-Department Store	5340
56.	National Biscuit	Biscuit Baker	2020
57.	American Metal Climax	Miscellanecus Metal	3380
58	Kennecott	Copper	3331
59.	American Smelting & Refining	Miscellaneous Metal	3350
60.	Pullman	Railroad Equipment	3740
61.	Clark Equipment	Material Handling Mach.	3530
62.	International Harvester	Agricultural Machinery	3520
63.	Joy Mfg.	Specialty Machinery	3560
64.	International Paper	Paper	2640
65.	Alcoa	Aluminum	3334
66.	Eaton, Yale & Towne	Auto Parts & Accessory	3780
67.	Borg-Warner	Miscellaneous	3790
68.	Otis Elevator	Miscellaneous	3510
69.	National Lead	Miscellaneous	2860
70.	Bethlehem Steel	Steel	3310
71.	National Steel	Steel	3310
72.	Gardner-Denver	Industrial Machinery	3560
73.	Rohm & Haas	Chemical	2850
74.	Johns-Manville	Building Material-Roofing	3275
75.	Ingersoll	Industrial Machinery	3560
76.	Goodyear	Tire & Rubber	3010
77.	Georgia-Pacific	Forest Product	2440
78.	Weyerhaeuser	Forest Product	2440
79.	Caterpillar	Construction Machinery	3530
80.	Timken	Miscellaneous	3590
81.	Sunbeam	Household Appliance	3680
82.	Deere	Agricultural Machinery	3520
83.	American Can	Container-Metal & Glass	3421
84.	Continental Can	Container-Metal & Glass	3421
85.	Consolidated Freightways	Trucker	3711
86.	Cincinnati Milling	Machine Tools	3542
87.	Babcock & Wilcox	Steam Generating Mach.	3525
88.	Square D	Electrical Equipment	3690
89.	American Standard	Building Mat.-Plumbing	3478
90.	Monsanto	Chemical	2850
91.	Burlington	Textile	2210
92.	Mohasco	Home Furnishing	2270
93.	Standard of California	International Oil	2930
94.	Texaco	International Oil	2930
95.	Jersey Standard	International Oil	2930
96.	Mobil	International Oil	2930
97.	Standard of Indiana	Domestic Oil	2950
98.	Gulf Oil	International Oil	2930
99.	Union Oil	Domestic Oil	2950
100.	Shell Oil	Domestic Oil	2950

for certain other groups of stocks—such as airlines and television, that could fit either growth or cyclical classes and soft drinks that could be either growth or stable—resisted classification. Finally, some investors believe that stocks of certain industries, such as construction or aerospace, may display unique price behavior because of their economic characteristics, thereby creating stock groupings independent of those originally hypothesized.

Statistical Analysis of Sample

Statistical analysis of the sample data had as its prime objective determining whether the three hypothesized stock classifications (growth, cyclical, stable) were in fact uncorrelated (independent) groups. In addition, this analysis promised to show whether stocks grouped into the assigned classifications and also to determine groupings for stocks not otherwise readily classified. Finally, the analysis was designed to indicate whether classification groups other than the three hypothesized (growth, cyclical and stable) were necessary.

The statistical analysis began with the calculation for each stock of monthly returns (i.e., price changes plus dividends received in the month) for the 108-month period from 1961-69. It was, however, not possible to work directly with these unadjusted returns. (Recall from the previous section that King's study of the magnitude of the market factor in explaining stock returns indicates that the returns of all stocks are, to some extent, affected by movements in the general market.) Over long market declines, virtually all stocks will show at least some decline; conversely, over long market rises, virtually all stocks will show at least some appreciation. As also noted, this tendency to move with the market gives rise to statistical correlation among all stocks. There was thus need to adjust the sample stock returns for a common relationship to the market in order to directly test whether additional common movement conformed to the growth, cyclical, and stable characteristics of stocks.

The following procedure was used to remove the market relationship from stocks in the sample. The S&P 425 industrial index was selected as representative of general market movements, and monthly returns for the index were calculated for the 1961-1969 period. Returns (R_i) for each stock in the sample were then regressed on the S&P 425 index (R_m) for the 1961-69 period. The form of the regression is the same as used for the single-index model and is shown as follows:

$$R_i = A_i + B_i (R_m) + C_i$$

The residual (C_i) represents the stock return that has been adjusted for the market relationship. Recall that one of the major specifications of the single-index model is that these residuals should be uncorrelated, that is E $(C_iC_j) = 0$—i.e., that the sole source of comovement among stocks is their relationship to the general market. Hence, a test for residual correlation according to growth, cyclical, and stable characteristics is in a sense a test of the validity of the single-index model specification of uncorrelated residuals.

Once returns had been adjusted for the market relationship, a coefficient of correlation was calculated between each stock in the sample and every other stock. Recall that the coefficient of correlation is a statistical measure of association between two stocks ranging in value between +1 and -1.[2] With 100 stocks in the sample and each

FIGURE 2

Matrix of Correlation Coefficients

Growth, Cyclical, and Stable Stocks

	GROWTH	CYCLICAL	STABLE
GROWTH	GROWTH STOCKS H+	L	L
CYCLICAL	L	CYCLICAL STOCKS H+	L
STABLE	L	L	STABLE STOCKS H+

2. Two securities with perfectly correlated return patterns will have a correlation of +1. Conversely, if the return patterns are perfectly negatively correlated, the correlation coefficient will equal -1. Two securities with uncorrelated (i.e., statistically unrelated) returns will have a correlation coefficient of zero. The correlation coefficient between returns for securities in this sample and the S&P 425 Market Index during the 1961-69 period was generally between 0.5 and 0.6.

being correlated with every other, the result is (100 x 100) = 10,000 correlation coefficients.

If stocks in fact group according to growth, cyclical, and stable characteristics, we would expect them to exhibit the pattern of high correlation within groups and low correlation across groups that is illustrated by the matrix of correlation coefficients in Figure 2. Specifically, the correlation coefficients within each of the classes—growth, cyclical and stable stocks, arranged along the diagonal of the matrix—should show high and positive values. These are identified within each of the groupings by the letter H and a plus sign. The correlation coefficients off the diagonal represent the correlation of stocks between groups (growth with cyclical, cyclical with stable, etc.), and are expected to be low in value. These are identified by the letter L.

Cluster Analysis

The test whether the sample stocks in fact showed the expected pattern is based on a statistical technique known as cluster analysis, which systematically examines the matrix of correlation coefficients.[3] The cluster analysis technique separates stocks into groups or clusters within which stocks are highly correlated, and between which stocks are poorly correlated. It is a stepwise process that involves (1) searching the correlation matrix for the highest positive correlation coefficient; (2) combining these stocks to reduce the matrix by one; and (3) recomputing the correlation matrix to include the correlation between the combined stock or cluster and the remaining stocks or clusters. This process continues in an iterative fashion until all positive correlation coefficients are exhausted or until (on the 99th pass) every stock is formed into a single cluster.

If the hypothesis that the 100 stocks in the sample could be categorized were correct and if the individual stocks were correctly classed, then by the 97th pass the three remaining groups should correspond to those hypothesized. Furthermore, all positive correlation coefficients should be exhausted by the 97th pass, thus terminating the process. This result would indicate, not only that the sample data can be explained by three independent groupings, but also that there is a low degree of correlation across stock groupings.

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3. The cluster technique used in this study is similar to one used by King, *op. cit.*, and termed a "quick and dirty" method of factor analysis. King viewed the routine as a method of exploration properly falling under the heading of "data analysis" rather than "inference," the results of which would be subject to testing and confirmation via other techniques. The cluster results in this study were, in fact, confirmed by other tests reported in an article by James L. Farrell, Jr., "Analyzing Covariation of Returns to Determine Homogeneous Stock Groupings," *Journal of Business*, April 1974, pp. 181-207.

Figure 3 provides a set of four diagrams showing the results of the cluster routine. The diagrams show the stage where pairs or groups of stocks joined, along with the value of the correlation coefficient at which they joined. Describing the cluster results required four diagrams, because an oil group emerged from the analysis in addition to the three groups originally hypothesized (the routine terminated on the 96th pass as no positive correlation coefficients remained at that stage). Actually, it had originally been expected that oils would group with the stable category of stocks, but the oil cluster failed to show a positive correlation with the stable cluster (or any other, for that matter).[4] No other independent groups emerged from the cluster analysis. The aerospace and building stocks—that might have been expected to exhibit independent behavior—clustered with the growth and cyclical groups respectively.

The number of stocks in each cluster is: growth, 31; cyclical, 36; stable, 25; and oil, 8. Stocks that had been given an a priori classification of growth, stable or cyclical actually clustered with their allocated groups. Naturally, all oil stocks clustered together. Those stocks that were not easily classified on an a priori basis—such as television, airlines and soft drinks—generally clustered with a group that could be accepted as reasonably appropriate. All group clusters appeared to contain highly intercorrelated stocks; final stocks or group of stocks clustering into individual groups did so at relatively high levels of positive correlation: 0.19, growth; 0.15, stable; 0.18, cyclical; and 0.27, oil. The final four groups were not positively correlated, as evidenced by the fact that the routine terminated on the 96th pass (positive correlation of 0.15 was the lowest positive correlation on the prior, 95th pass).

Index Procedure

To determine the degree of positive correlation of stocks within each of the four groupings of growth, cyclical, stable and oil stocks as well as the extent to which stocks were uncorrelated with those of other groupings, the author developed first a rate of return (adjusted for general market effects) for each of the four stock groupings. Stocks that clustered into the four groups were formed directly into four monthly indexes composed of (1) 31 growth stocks, (2) 25

4. Recall that King's results indicated that the oil industry was also the only one that showed no positive correlation with any of the other five industries analyzed and is further evidence that the oil group is sufficiently unique to be considered an independent category.

FIGURE 3
CLUSTER DIAGRAM

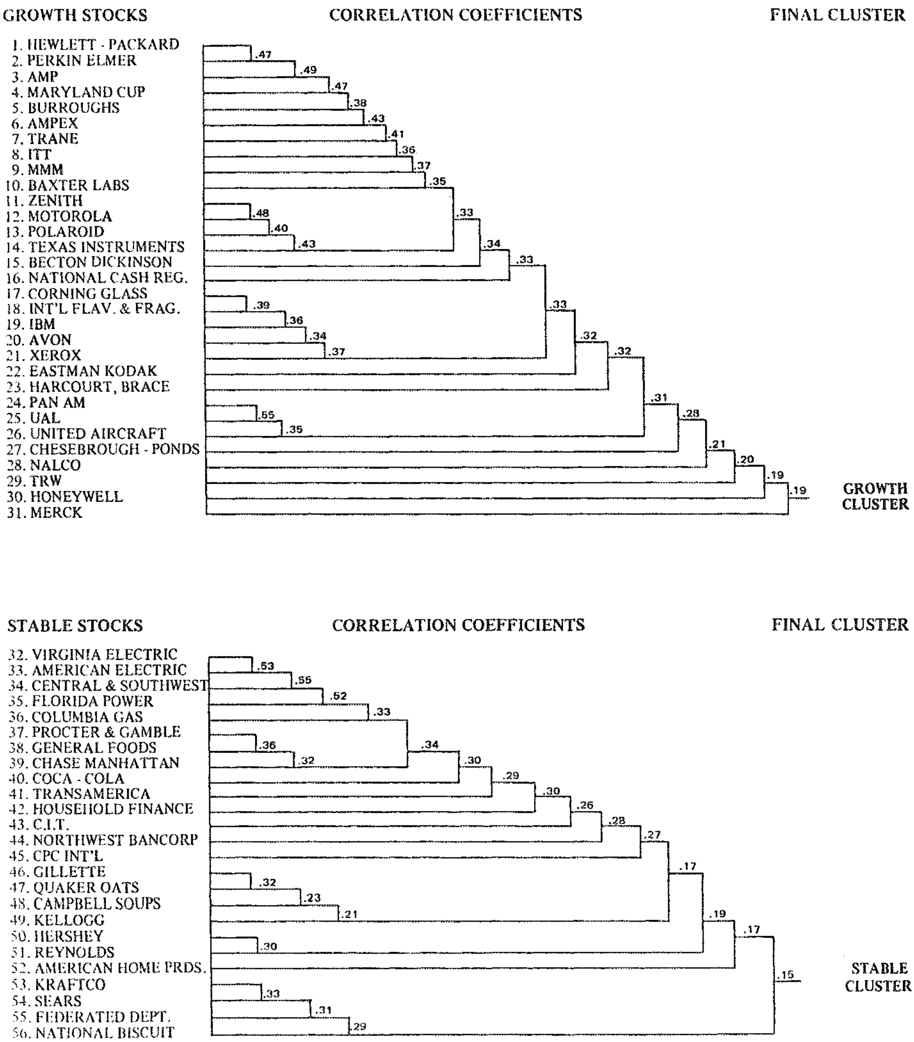
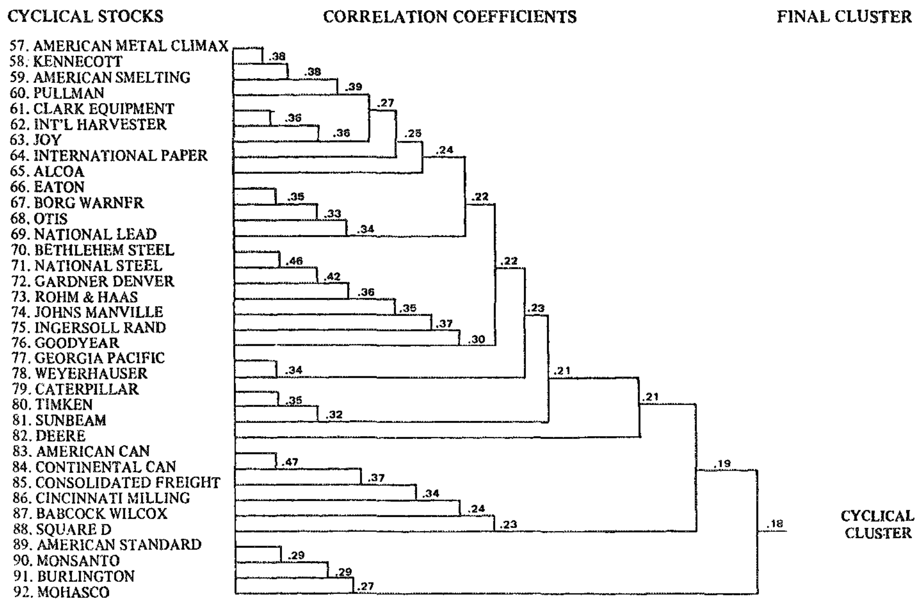
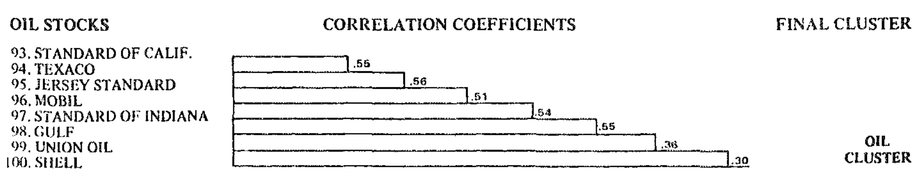


FIGURE 3 (Cont'd)





stable stocks, (3) 36 cyclical stocks, and (4) 8 oil stocks. The author then averaged the adjusted return of each stock in the index month by month—to provide, for example, a series of 108 monthly growth index returns for the 1961-69 period.

Between the adjusted returns of each stock in the sample (100 stocks) and each index (four indexes), the author then calculated 400 different correlation coefficients. If the four groupings of growth, cyclical, stable and oil stocks represent homogeneous groups, one would expect the correlation coefficient of a stock with its corresponding index to be positive and significantly different from zero. In addition, one would expect its correlation with the other three indexes to be limited. This pattern is implied by the previously described expectation for the matrix of correlation coefficients of the grouped stocks (see Figure 2).

Figure 4 shows an average value of the correlation coefficients for the (1) 31 growth, (2) 36 cyclical, (3) 25 stable, and (4) 8 oil stocks

FIGURE 4

Matrix of Residual Correlation Coefficients
Averages of Stocks and Four Stock Indexes
(1961 - 1969)

	GROWTH INDEX	CYCLICAL INDEX	STABLE INDEX	OIL INDEX
GROWTH AVERAGE	.43	-.05	-.05	-.17
CYCLICAL AVERAGE	-.06	.38	-.04	-.02
STABLE AVERAGE	-.05	-.03	.37	-.07
OIL AVERAGE	-.26	-.03	-.12	.64

with the corresponding index as well as with the other three indexes (this 4 x 4 matrix thus summarizes the full 4 x 100 correlation matrix of stocks and indexes). The average values of the correlation coefficients of stocks with the corresponding index were all positive and highly significant (correlations of 0.19 or higher occur by chance only one time in 20). At the same time, the average values of the correlation coefficients of stocks with other indexes indicated a low degree of correlation.

In addition, inspection of the complete correlation matrix of stocks and indexes (complete matrix not specifically shown) indicated that each stock included in the group average was positively correlated with its respective index at a statistically significant level. Specifically, all growth stocks were positively and significantly correlated with the growth index, as were stable stocks with the stable index, cyclicals with the cyclical index, and oils with the oil index. Only six stocks showed significantly positive correlation with an index other than the one assigned and in each case the stock was more highly correlated with its assigned index. With the exception of these six stocks, all others displayed either negative correlation or less than significant positive correlation with indexes other than the index for their own groupings. It is hard to avoid the conclusion that growth, cyclical, stable, and oil stocks do in fact represent homogeneous groupings.

Significance of Group Effects

One can use regression analysis to determine the percentage of the observed variations in return on the individual stock explained by such systematic factors as market, or growth, cyclical, stable, and oil. This study used the S&P 425 Index to represent the general market factor, while the previously described indexes of adjusted returns were used to represent growth, cyclical, stable, and oil factors. The coefficient of determination (R^2) from the regression is an estimate of the percentage of realized returns of individual stocks explained by systematic factors.

The average R^2 for all stocks in the sample 100 with that of the market over the 1961-69 period was approximately 30 percent, which is consistent with the figures in other studies (including King's) of the postwar period. The average R^2 from the regression of the return of the individual stock on the adjusted return of the corresponding index (i.e., growth stock on growth index; cyclical on cyclical index; stable on stable index; and oil on oil index) for the same 1961-69 period was approximately 15 percent. These two systematic factors combined thus account for approximately 45 percent of the realized return of an individual stock, with the remainder attributable to the industry factor or effects unique to the individual security.[5]

As this completes the test of the stock grouping hypothesis, it would be useful at this point to emphasize the main findings of this section. It is that four groupings of stocks (growth, cyclical, stable, and oil) are homogeneous in the sense of being highly correlated within groupings and not significantly correlated with other stock groupings. These four groupings in turn represent a broader-than-industry factor additional to the market and industry factors previously determined by King. Correspondingly, these groupings that are homogeneous with respect to collinearity characteristics provide a suitable means for developing inputs to the multi-index portfolio selection model.

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5. Recall that King showed that the industry factor accounted for 10 percent of the realized return of common stocks during the 1927-60 period. Adding 10 percent for industry effects (under the assumption that this effect has continued to maintain a reasonable degree of stability over time) to the 45 percent of the other two systematic factors indicates that these three effects may explain 55 percent of the realized return of common stocks. This would in turn indicate that effects unique to a security are somewhat less than 50 percent.

V. Test of Relative Performance of Multi-Index Model

This section describes a test of the performance of the multi-index model using indexes constructed from homogeneous groups of growth, cyclical, stable and oil stocks. The test includes an analysis of the facility of both the multi-index and single-index models in generating ex-ante efficient sets of portfolios as well as a comparison of the performance of these efficient portfolios to that of the S&P 500 and a set of mutual funds over an ex-post period. The final part of the paper analyzes the stability of input relationships to the model. This analysis is important in indicating the extent to which historic data can be used in the practical application of the model.

Analysis of Index Model Specifications

The existence of stock groupings homogeneous with respect to collinearity characteristics has two immediate implications for portfolio selection models. First and most obviously, these groups provide the basis for constructing indexes with desirable collinearity characteristics that can be used as inputs in the multi-index model. Second and less directly, the findings imply a potential deficiency of the single-index model in generating efficient sets of portfolios. The way this potential deficiency arises can be illustrated as follows.

The effectiveness of an index model, whether single-index or multi-index, is directly related to the ability of the model to reproduce the true correlation relationships among securities. This facility is, in turn, related to the degree to which each model fulfills a specification referred to as uncorrelated residuals. In the case of the single-index model, one of the basic assumptions is that the sole source of co-movement among stocks is due to a general market relationship. This assumption implies in turn that there should be no correlation among the residuals of stocks, once the general market effect has been removed—i.e., that $E\ (C_iC_j) = 0$.

In the previous section the initial steps in testing for homogeneous stock groupings were to remove the general market effect from stock returns, and then to test for patterns of correlation among the residuals. The test results showed that the residuals of stocks in the sample were correlated according to the growth, cyclical, stable and oil characteristics of stocks. This finding not only confirmed the existence of a broader-than-industry factor for grouping stocks but also indicated a direct violation of the specification of uncorrelated residuals in the single-index model.

In order to evaluate the extent to which the single-index model violates the specification of uncorrelated residuals, as well as to appraise the adequacy of the four-index model in meeting this specification, we measured the degree of correlation within the residual matrices of both models. We derived (1) a residual correlation matrix for the single-index model by regressing each of the 100 stocks in the sample against the S&P 425 and correlating the residuals, and (2) a residual correlation matrix for the four-index model by a multiple regression of each of the 100 stocks against the four indexes of growth, cyclical, stable, and oil stocks respectively, and correlating the residuals.

Table 2 shows the distribution of the correlation coefficients of the residuals from the correlation matrices of the single-index and four-index regression models. The residual correlations from the four-index model cluster more closely about zero than the residual correlations from the single-index model: 15.6 percent of the residual correlation coefficients generated by the single-index model were statistically significant at 0.05 level—whereas only 6.5 percent of the coefficients generated by the four-index model were significant at that level. In addition, the percentage of statistically significant correlation coefficients generated by the single-index model was greatly in excess

Table 2

Distribution of Residual Correlation Coefficient from Simple and Multiple Regressions

Residual Correlation		Simple Regression	Multiple Regression
		(%)	(%)
Over	.30	3.00	.50
.30	.20	6.50	2.25
.20	.10	18.00	14.50
.10	.00	25.00	29.00
.00	-.10	26.50	32.00
-.10	-.20	15.00	18.00
-.20	-.30	4.50	3.50
-.30	Less	1.50	.25

of that expected by chance, whereas the percentage generated by the four-index model was not.[1]

1. These results in regard to the single-index model are consistent with those shown by Cohen and Pogue in Table 2 of their study where approximately 15 percent of the correlation coefficients were significantly different from zero. On the other hand, results of this study are in marked contrast to those shown in the same table in regard to the multi-index model. The Cohen and Pogue table indicated a wider dispersion of correlation coefficients as well as more than twice as many or approximately 15 percent significantly different from zero.

The extraordinary degree of correlation among the residuals from the single-index model provides further evidence that the specification of uncorrelated residuals for this model is seriously violated. This violation results from the single-index model's failure to incorporate systematic effects among securities other than those due to a common relationship with a general market factor. As a result, it is likely that the relationship among securities specified by the single-index model will differ substantially from the true correlation relationships.

In contrast, the residuals in the four-index model are virtually uncorrelated. This model was more effective than the single-index model in incorporating the major systematic elements explaining the cross-sectional structure of security returns over the 1961-69 period of analysis. The relationship among securities implied by the four-index model more closely approximates the true correlation matrix than that implied by the single-index model.

Ex-Ante Correlation Matrices

In order to appraise the effectiveness of the single-index and four-index models in reproducing the cross-sectional structure of security returns for the 1961-69 period, the implied correlation matrices of these models were compared to the true matrix as developed by the Markowitz formulation. The S&P 425 was used as an input to the single-index model while four indexes of 31 growth, 25 stable, 36 cyclical, and 8 oil stocks were used as inputs to the multi-index model. The true correlation matrix of security returns was constructed by using correlation relationships among securities over the 1961-69 period.

Differences were then taken between the true correlation coefficients and the equivalent correlation coefficients generated by each index model and tabulated into frequency distributions of differences as shown in Table 3 below:

Table 3

Distributions of Correlation Coefficient Error

Correlation Coefficient Error		Relative Frequency Single-Index Model (%)	Relative Frequency Multi-Index Model (%)
Over	.20	3.63	.54
.20	.10	15.82	9.35
.10	.00	30.10	37.69
.00	-.10	31.09	41.86
-.10	-.20	15.66	9.75
-.20	Less	3.80	.79

Note that the distribution of differences for the multi-index model was tightly centered about zero, with approximately 80 percent of the differences clustered within a range of ± 0.10. On the other hand, only a little over 60 percent of the coefficient differences for the single-index model were distributed within the ± 0.10 range. These results indicate that the multi-index model was, in fact, superior to the single-index model in representing the true ex-ante correlation matrix.[2]

Ex-Ante Efficient Portfolios

The facility for more accurately representing the correlation relationships among securities should, in turn, allow the multi-index model to generate a set of efficient portfolios superior to those generated by the single-index model. In order to test the relative performance of the two models in this regard, a set of portfolios for each model was generated and compared with respect to location of the ex-ante efficient frontiers and composition of portfolio. Input data for the models were developed by assuming that the performance over the 1961-69 period of the 100 stocks in the sample, the S&P 425, and the four stock group indexes was representative of expected values at the end of the period.

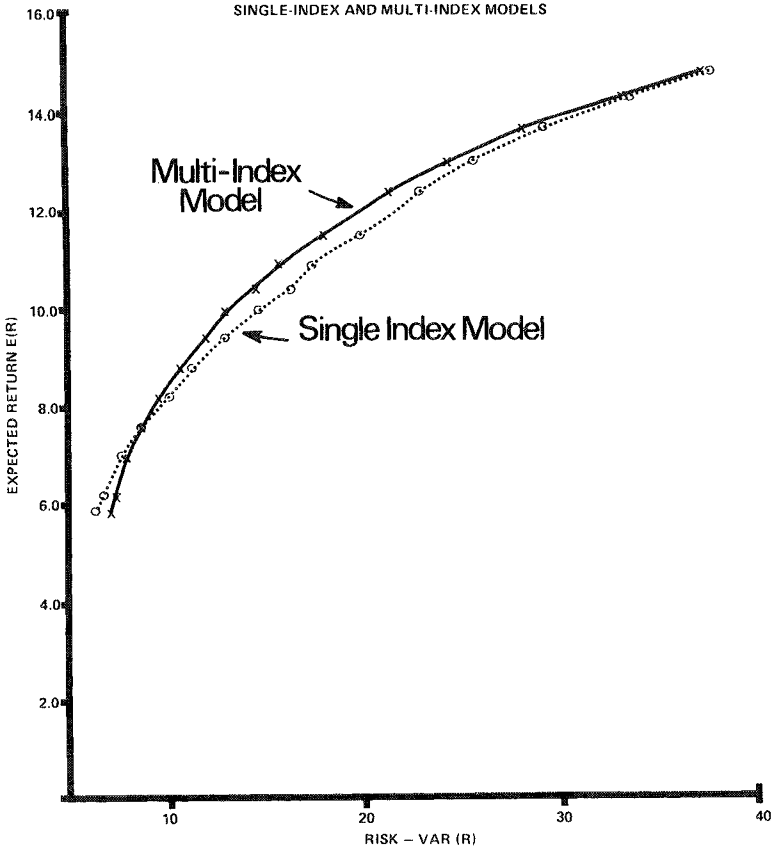
More specifically, the expected values of the S&P 425 market index and the four indexes of growth, cyclical, stable and oil stocks were assumed to be an average of monthly returns over the 1961-69 period, while the variability of these indexes was assumed to be equal to that computed for this same period. The covariability of the four stock indexes was also assumed to be equal to that computed for the 1961-69 period. Finally, the specific return (A_i), measure of responsiveness (B_i), and variance of residual return (Var C_i) of individual securities with the indexes (either S&P 425 or pertinent stock group)

2. These results showing the superiority of the multi-index model over the single-index model in representing the true ex-ante correlation matrix are in marked contrast to those presented by Cohen and Pogue in their study. As described in Section III of this study, the authors did not ensure that the residuals (after removal of the general market factor) of the industry groups used as indexes for the multi-index model were highly intercorrelated and at the same time noncollinear across groups. In contrast, the residuals of groups of stocks used in this study as indexes for the multi-index model did meet these collinearity requirements and thereby insured that systematic comovement among stocks would be better specified than in the case of the single-index model.

were derived from regressions of monthly stock and index returns over the 1961-69 study period.[3]

Figure 5 is a risk-return diagram (vertical axis is return and horizontal axis is risk) showing the ex-ante efficient frontiers of portfolios generated by both the single-index and multi-index models.

FIGURE 5
EX-ANTE EFFICIENT FRONTIER
SINGLE-INDEX AND MULTI-INDEX MODELS



3. While historic values for the specific return (A_i) of individual stocks with the market index or stock group index were calculated, this test assumed that these values were zero for the following reasons. First, portfolio selection models tend to select those securities with high positive specific returns (A_i values), thus resulting in portfolios heavily concentrated in those securities unless limited by model constraints (i.e., maximum of five percent of any single issue). Second, analysis of the stability of input relationships to the models in the latter part of this section demonstrates that specific return values (A_i values) tend to be highly unstable or not significantly different from zero. Establishing zero as a best estimate of the future value of this input thus avoids this problem of instability as well as circumvents the need to specifically establish constraints on the models.

Note that the efficient frontier for the multi-index model is above and to the left of the single-index model frontier over all but extremely high and low risk-return levels. The multi-index model provides the same return with less risk, thus dominating the single-index model over a wide range of returns.[4] This dominance of the multi-index model over the single-index model confirms the expected superiority of the multi-index model in generating efficient portfolios.

Table 4 shows, at equivalent levels of return, the composition of efficient portfolios generated by the single-index and multi-index portfolio selection models. More specifically, this table shows the number of securities, as well as the maximum percentage represented by a single security, in each of the efficient portfolios generated by the two models. In addition, the table shows the percentage that each of the four homogeneous groups of growth, cyclical, stable, and oil stocks represents in each of the two sets of efficient portfolios. Data in this table thus provides a means of determining how well the two models provide portfolios that are well diversified.

The number of securities and the percentage represented by the largest single security indicates that the portfolios generated by both index models are well diversified with respect to these two measures.[5] More specifically, note that except for the very highest levels of return (at highest level of return each portfolio contains only one security promising the highest return), both sets of efficient portfolios contain at least 16 securities. Researchers have determined that randomly generated portfolios of 16 or more securities are fairly well diversified.[6]

4. It should be noted that both index models understate the variances of the portfolios, having been computed by the "reduced" covariance matrices implicit in these models. Cohen and Pogue in their study calculated the variances of efficient portfolios generated by the index models using the true Markowitz formulation and compared these results to those using the reduced variances of the index models. Cohen and Pogue found that the reduced variances of the single-index model understated the true variances to a greater extent than in the case of the multi-index model. The relationship shown in Figure 5 may thus understate the dominance of the multi-index model over the single-index model.

5. Cohen and Pogue imposed upper bound constraints on the proportion that an individual security could represent in an efficient portfolio to ensure that portfolios were well diversified. As noted earlier, assumption of a zero value for the specific return input (A_i value) was expected to circumvent the need for constraints. This, as illustrated by the results of the test, proved to be the case.

6. Lawrence Fisher and James Lorie, "Some Studies of Variability of Returns on Investments in Common Stocks," *Journal of Business*, April 1970, pp. 99-134.

Within the reasonably attainable return range of 1.14 percent and 0.62 percent (monthly returns), the average number of securities in the multi-index portfolio was 33, while the average number of securities in the single-index portfolio was 82. Within the return range from 1.14 percent to .62 percent the average fraction of the multi-index portfolio constituted by the largest single security was 10.2 percent; the average fraction of the single-index portfolio was 5.0 percent.

While both index models developed sets of portfolios that were well diversified, the multi-index model is more efficient in developing this diversification. In particular, the multi-index model generates lower risk portfolios at equivalent levels of return using fewer securities than the single-index model. The superiority of the multi-index model results from its facility in diversifying across the homogeneous (with respect to collinearity characteristics) groupings of growth, stable, cyclical, and oil stocks. Note from Table 4 that proportions of stocks in the single-index model portfolios tend to be concentrated in one or two of the homogeneous groupings, whereas the multi-index model portfolios are more evenly distributed across the four groupings over a wide range of returns.

Ex-Post Performance

The preceding tests of model specification, correlation matrices, and location and composition of efficient portfolios, based on the period 1961-69, all indicate that before the fact the multi-index model was superior to the single-index model. To assess the relative capability of the two models ex-post, the author evaluated the performance of the efficient portfolios generated by the two models over the five years 1970-74. The performance of these portfolios was also compared to that of the S&P 500 and a set of large all-stock mutual funds.

Before proceeding with this analysis, the difference in market environment between the ex-ante (1961-69) and ex-post (1970-74) periods will be considered. Note from Figure 6 that Treasury bills averaged 3.9 percent while the return on the S&P 500 was 8.6 percent; the risk premium for stocks over the 1961-69 period was a positive 4.7 percent. In contrast, the data in Figure 7 indicate a negative risk premium for stocks of close to 7 percent over the 1970-74 period. Treasury bills averaged 5.8 percent while the S&P 500 was down an average of one percent over the period.

Table 4

Composition of Efficient Portfolios

	Single Index Model						Multi-Index Model					
			Homogeneous Stock Groupings						Homogeneous Stock Groupings			
Return	Stocks in Portfolio	Largest Single Stock	Growth	Stable	Cyclical	Oil	Stocks in Portfolio	Largest Single Stock	Growth	Stable	Cyclical	Oil
		(%)	(%)	(%)	(%)	(%)		(%)	(%)	(%)	(%)	(%)
1.34	1	100	100	–	–	–	1	100	100	–	–	–
1.33	2	76	100	–	–	–	2	91	100	–	–	–
1.32	3	47	77	–	23	–	3	53	76	–	24	–
1.29	5	25	52	25	23	–	4	33	42	33	25	–
1.28	6	26	49	26	25	–	5	33	41	33	26	–
1.27	8	25	49	25	26	–	7	32	39	32	29	–
1.25	10	22	47	22	31	–	8	30	34	30	36	–
1.24	11	21	46	21	33	–	10	29	34	29	37	–
1.23	14	19	47	19	34	–	13	27	36	27	37	–
1.22	16	17	49	17	34	–	14	26	36	26	38	–
1.20	19	15	52	15	33	–	15	25	37	25	38	–
1.19	21	14	53	14	33	–	16	24	37	24	39	–
1.18	24	13	54	13	33	–	17	23	37	24	39	–
1.17	26	12	53	12	35	–	18	22	38	24	38	–
1.16	28	12	53	12	35	–	19	22	37	24	38	1
1.14	33	9	55	9	36	–	20	19	36	23	37	4
1.13	34	9	53	9	38	–	22	18	37	22	37	4
1.12	36	8	54	8	38	–	22	18	37	22	37	4
1.11	38	7	53	8	39	–	22	17	37	22	36	5
1.08	46	6	52	8	40	–	22	15	37	21	34	8
1.06	51	5	52	8	40	–	21	14	37	20	33	10
1.05	58	5	48	8	44	–	23	13	36	20	33	11
1.04	59	5	47	9	44	–	24	12	36	20	32	12
1.03	61	4	49	9	41	1	27	11	36	21	31	12
1.01	70	4	44	10	45	1	28	10	36	20	31	13
1.00	74	4	44	11	44	1	29	10	36	21	30	13
.98	77	4	41	12	45	2	30	9	35	22	29	14
.96	82	3	39	14	44	3	32	8	35	22	28	15
.95	87	3	38	15	44	3	32	8	35	23	27	15
.94	90	3	37	16	44	3	33	8	34	23	27	16
.93	91	3	35	17	44	4	33	7	34	23	27	16
.91	95	3	32	17	45	6	32	7	34	24	25	17
.90	96	3	31	18	45	6	32	7	34	24	25	17
.89	98	3	32	20	41	7	33	7	34	24	24	18
.88	99	3	31	22	40	7	34	7	34	24	24	18
.87	99	3	30	25	38	7	34	7	34	24	24	18
.86	99	2	29	27	37	7	34	7	33	24	24	19
.85	99	2	28	28	36	8	35	7	33	25	23	19
.84	99	2	27	29	36	8	36	7	33	26	22	19
.83	99	2	26	29	36	9	36	7	32	27	21	20
.81	99	2	24	31	36	9	39	7	32	27	21	21
.80	99	3	22	32	36	10	39	7	31	27	21	21
.79	99	3	20	34	36	10	39	7	31	27	21	21
.78	99	4	19	34	36	11	39	7	30	28	21	21
.77	99	4	18	35	35	12	39	7	30	28	21	21
.76	99	4	18	34	35	13	39	7	29	29	21	21
.74	99	5	17	35	35	13	38	7	29	29	21	21
.73	99	5	16	35	35	14	38	7	28	30	21	21
.71	99	5	16	35	35	14	38	8	27	31	21	21
.70	96	6	15	37	33	15	39	8	26	32	21	21
.69	95	6	14	37	34	15	39	10	26	33	20	21
.68	94	7	13	39	33	15	38	11	26	33	20	21
.66	89	7	12	40	32	16	38	12	24	33	23	20
.65	86	8	12	41	31	16	37	13	24	34	21	21
.64	80	8	10	42	31	17	36	14	24	35	20	21
.63	75	9	9	43	31	17	35	15	24	35	20	21
.62	71	9	9	44	29	18	33	16	24	36	20	20
.60	63	11	7	46	28	19	31	18	21	37	21	21
.59	57	12	7	48	25	20	28	20	20	39	21	20
.58	55	14	6	50	24	20	28	21	20	39	21	20
.57	53	14	6	50	24	20	27	22	19	40	21	20
.55	47	16	6	51	23	20	26	24	18	41	21	20
.54	40	16	5	52	23	20	25	25	17	42	21	20
.52	38	18	5	53	22	20	25	28	15	43	22	20
.49	31	25	4	55	22	19	24	31	13	46	22	19
.48	25	27	4	54	23	19	22	33	12	47	22	19

FIGURE 6

Risk Return Relationship
(1961 - 1969)

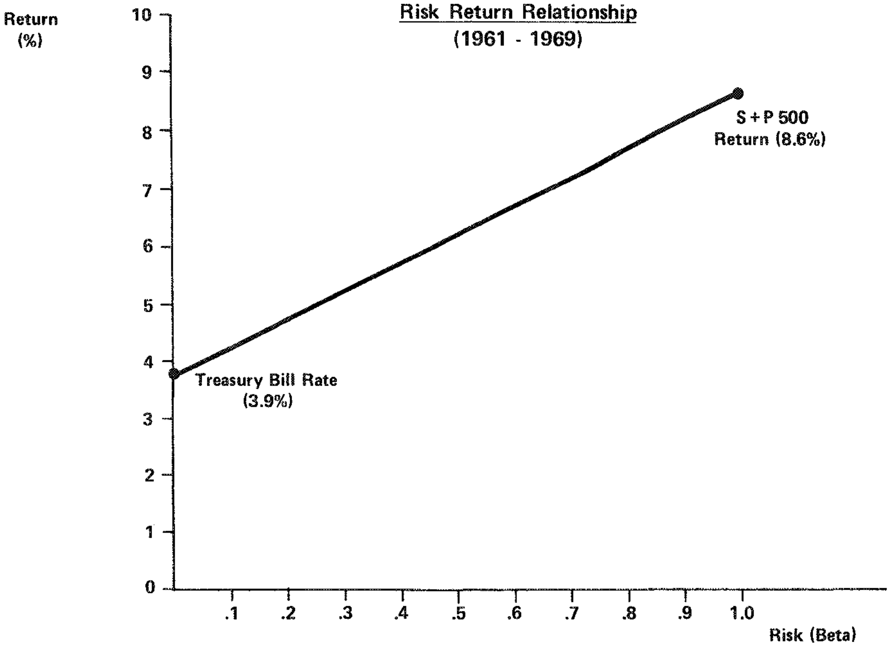
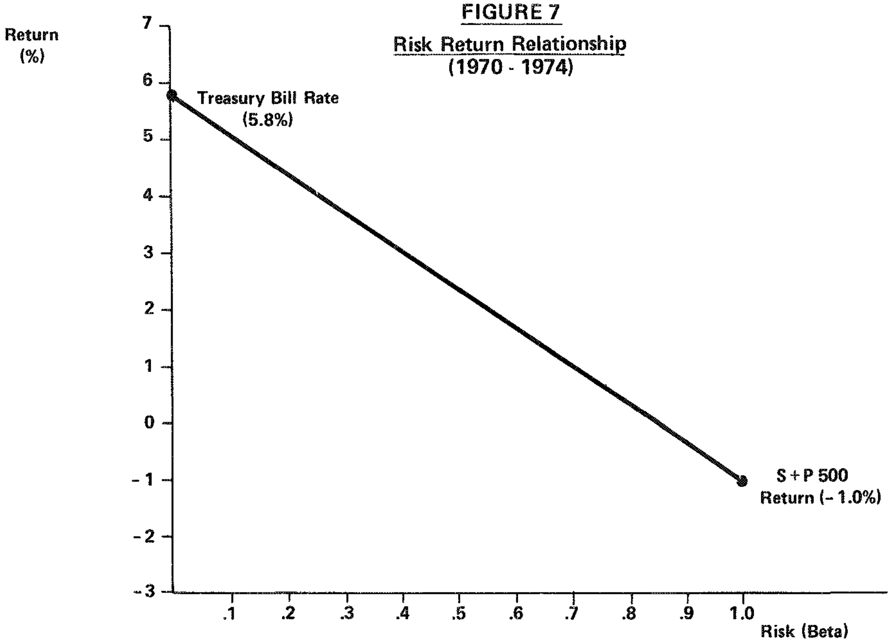


FIGURE 7

Risk Return Relationship
(1970 - 1974)



The five years (1970-74) might be characterized as a long bear market in contrast to the 1961-69 period that was a long bull market. It is important to bear in mind this negative ex-post market relation when evaluating performance. In contrast to normal bull market periods, higher than market risk portfolios would be expected to show greater than average losses. At the same time, lower than average market risk portfolios would be expected to show less than average losses (and perhaps some gains).

Table 5 shows performance statistics for the S&P 500 and 33 large all-stock mutual funds (all classified as growth or income-growth funds by Weisenberger). Specifically, the table shows the average

Table 5

Mutual Fund Performance
(1970-1974)

Fund	Average Return	Standard Deviation	Alpha	Beta	R^2
Affiliated	.7	4.9	2.0	.93	.86
American Mutual Fund	1.5	4.7	2.2	.89	.87
Anchor Growth Fund	-10.6	6.5	-9.0	1.24	.88
Vance, Sanders Investors	.2	3.8	-.2	.74	.89
Broad Street Inv. Corp.	1.0	4.7	1.4	.94	.95
Chemical Fund	1.1	5.1	1.2	.98	.90
Colonial Fund	-.5	4.2	-.5	.80	.86
Delaware Fund	-3.2	5.3	-2.2	.99	.84
Dividend Shares	-.4	4.4	-.3	.88	.98
Dreyfus Fund	-4.3	5.2	-3.2	.98	.88
Eaton & Howard Stock	-4.5	5.4	-3.7	1.07	.94
Fidelity Capital	-6.2	6.2	-4.8	1.20	.91
Fidelity Fund	-.4	4.8	.5	.95	.95
Fundamental Investors	-4.9	4.7	-4.6	.92	.93
US Life Common Stock Fund	1.0	4.1	1.1	.78	.86
Hamilton Funds Ser. H-DA	-2.8	5.0	-1.7	.98	.93
Invest. Co. of America	-.1	4.9	.8	.96	.93
Investors Stock	-3.3	5.3	-2.2	1.06	.96
Investors Vari. Payment	-4.9	6.0	-4.6	1.17	.92
Johnston Mutual Fund	-1.4	5.7	-1.2	1.08	.90
Keystone K-2	-2.1	6.2	-2.1	1.14	.81
Keystone S-3	-4.6	6.3	-3.7	1.18	.84
Keystone S-4	-11.5	8.4	-11.7	1.50	.77
Mass. Inv. Growth Stock	-3.3	5.9	-2.8	1.13	.90
Mass. Inv. Trust	-3.4	4.8	-2.7	.94	.95
National Investors Corp.	-2.8	6.0	-2.4	1.16	.91
National Growth	-8.0	5.7	-8.2	1.09	.89
Price (Rowe) Growth Stock	-3.7	6.1	-3.1	1.18	.91
State Street Inv. Corp.	-2.7	5.6	-1.6	1.08	.90
Technology Fund	-3.1	5.5	-1.5	1.06	.89
United Accumulative Fund	-3.9	5.7	-2.2	1.13	.96
United Science Fund	-7.1	5.7	-5.5	1.13	.93
Wash. Mutual Investors	.7	4.9	1.7	.92	.85
Average Fund	-2.95	5.38	-2.27	1.04	0.90
S&P 500	-1.0	4.9	0	1.00	1.00

return earned by the fund or S&P 500 as well as the standard deviation of return over the 1970-74 period. It also shows standard risk adjusted performance statistics derived from a regression of the monthly return of the fund versus that of the S&P 500 over the 1970-74 period.[7] The beta of the fund provides a measure of the risk of the fund, while the alpha value is a measure of risk adjusted performance (positive value indicates above average performance while a negative value indicates below average performance). The table also shows the coefficient of determination (R^2) from the regression of fund and S&P 500 return to provide a measure of diversification of the fund (a value of 1.00 would indicate perfect diversification).

Note at the bottom of the table that the S&P 500 showed a negative return of 1.0 percent while funds lost an average of close to 3.0 percent over the 1970-74 period. Funds on average showed a slightly greater risk than the S&P 500, as measured by the standard deviation and an average beta of 1.04. The risk (beta) adjusted performance of the funds was generally below average, as evidenced by an average negative alpha of over two percent. Finally, note that the average R^2 value was 0.90, indicating that the funds incurred diversifiable risk.[8]

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7. Note that risk-adjusted performance figures are derived by using the single-index model specification, which only considers the effect of general market movements on performance. Results of this study however indicate the need to also adjust for the effect of growth, cyclical, stable and oil characteristics in a more complete performance evaluation model. For a more complete discussion of this problem, refer to James L. Farrell, Jr., "Homogeneous Stock Groupings: Implications for Portfolio Management," *Financial Analysts Journal*, May-June, 1975.

8. For purposes of further comparison, the performance of an equally weighted portfolio of the 100 sample stocks was compared to that of the S&P 500 for the 1970-74 period. This portfolio showed an average loss of 1.0 percent that was exactly in line with the loss of the S&P 500 over the period. The beta of this portfolio was somewhat higher than the S&P 500 at 1.07 and the risk adjusted performance (alpha) was virtually zero.

Table 6 shows the same sort of performance data for portfolios generated by the multi-index and single-index models. There are 20 portfolios shown for each model; these are representative of the ex-ante efficient frontier of portfolios for each model. Portfolios at extremely high or low ex-ante risk-return levels were eliminated from the ex-post comparison. These portfolios are generally not of interest as they contain too few securities. Also, the elimination reduces to a more manageable level the number to be analyzed.

The portfolios are arrayed from top to bottom according to their ex-ante risk. The resulting ranking of the portfolios of both models conforms almost perfectly to their ex-post risk, with portfolios with a high ex-ante risk displaying a higher realized standard deviation or beta than portfolios of lower ex-ante risk. For both models high-risk portfolios showed negative returns while low-risk portfolios realized positive returns in keeping with the bear market in the period 1970-74.

Measuring first the risk (beta) adjusted performance of the single-index model, note that the alphas for these portfolios are generally positive but not significantly different from zero, as evidenced by low "t statistics." It thus seems reasonable to conclude that the single-index model portfolios performed approximately in line with the S&P 500 (on a beta adjusted basis). Furthermore, it appears that the single-index model may have outperformed, and certainly did no worse, than the mutual funds, which on average recorded a negative beta adjusted performance.

At the same time, the multi-index model appears to have been superior to all three—S&P 500, mutual funds, and the single-index model. More specifically, a majority of the risk (beta) adjusted performance statistics (alphas) were positive.

In addition, the multi-index model portfolios were better diversified than the single-index portfolios and the mutual funds. The R^2 for the multi-index model portfolios were generally larger than the 0.90 mutual fund average. Within the beta range of 0.90 and 1.20 the R^2 averaged around 0.95 and was generally higher than comparable single-index model portfolios and well above those of the mutual funds. As a matter of fact, the R^2 values of 0.97 for the two multi-index model portfolios were higher than that of any single mutual fund in the sample.

To illustrate further the performance characteristics of the multi-index model, Table 7 specifically shows the composition of a multi-index model portfolio from the previous table (beta = 1.00; alpha = 2.0; R^2 of 0.97). Note that this portfolio contains 38 securities, with the largest single holding comprising only 7.2 percent of the total. This was substantially less than the 80 or more securities required by the single-index model to generate a high degree of diversification as well as less than the number typically held by a mutual fund. The

Table 6

Portfolio Selection Model Performance (1970-1974)

Multi-Index Model

Avg. Return	Std. Dev.	Alpha	t-Stat	Beta	R^2
-3.9	6.5	-1.3	-0.37	1.24	0.89
-3.1	6.3	-0.6	-0.22	1.22	0.91
-2.0	6.1	0.3	0.13	1.20	0.93
-2.1	6.0	0.2	0.07	1.19	0.94
-1.5	5.8	0.6	0.28	1.15	0.95
-0.7	5.6	1.1	0.58	1.11	0.96
-0.5	5.5	1.1	0.64	1.10	0.96
-0.1	5.4	1.3	0.79	1.07	0.96
0.3	5.2	1.5	0.97	1.04	0.96
0.5	5.1	1.7	1.14	1.03	0.97
1.0	5.0	2.0	1.36	1.00	0.97
1.2	4.9	2.0	1.31	0.97	0.96
1.8	5.0	2.7	1.65	0.99	0.96
1.8	4.8	2.4	1.30	0.94	0.94
1.9	4.9	2.6	1.37	0.96	0.94
2.5	4.6	2.7	1.17	0.89	0.90
2.9	4.6	3.0	1.24	0.88	0.89
3.4	4.5	3.3	1.21	0.85	0.85
3.6	4.5	3.4	1.14	0.83	0.82
4.0	4.5	3.6	0.99	0.79	0.75

Single Index Model

Avg. Return	Std. Dev.	Alpha	t-Stat	Beta	R^2
-4.0	7.1	-0.9	-0.19	1.31	0.83
-3.6	7.1	-0.4	-0.10	1.32	0.84
-2.9	6.8	0.02	0.01	1.29	0.86
-4.0	7.1	-0.8	-0.19	1.32	0.86
-3.4	6.8	-0.5	-0.12	1.28	0.88
-3.1	6.7	-0.2	-0.05	1.28	0.88
-2.8	6.6	0.01	0.00	1.26	0.89
-2.6	6.5	0.1	0.02	1.25	0.90
-2.4	6.3	0.1	0.05	1.22	0.91
-1.7	5.9	0.3	0.13	1.15	0.92
-0.8	5.7	1.0	0.44	1.12	0.94
-0.8	5.6	1.0	0.50	1.11	0.95
-0.7	5.6	1.0	0.52	1.10	0.95
-0.4	4.7	1.0	0.58	0.94	0.95
0.6	4.8	1.2	0.64	0.94	0.94
0.9	4.6	1.3	0.68	0.91	0.94
1.1	4.8	1.5	0.63	0.92	0.90
1.9	4.7	2.0	0.66	0.87	0.84
1.8	4.7	1.7	0.50	0.84	0.79
2.1	4.7	1.8	0.48	0.82	0.74

Table 7

Portfolio Composition – Multi-Index Model
(Beta = 1.00; Alpha = 2.0; R^2 -.97)

Stock	Portfolio Weight
Alcoa	1.5%
American Can Co.	2.9
American Home Products	2.3
Borg Warner Corp.	2.6
Burlington Inds. Inc.	2.2
Chesebrough Ponds Inc.	1.2
Columbia Gas Systems Inc.	5.0
Campbell Soup Co.	0.6
Deere & Co.	1.2
Eastman Kodak Co.	7.2
Federated Dept. Stores	5.2
Gulf Oil Corp.	3.2
Georgia Pacific Corp.	0.8
Gillette Co.	1.3
Goodyear Tire & Rubber	3.7
Honeywell Inc.	1.6
Int'l Business Machines	4.5
Int'l Paper Co.	3.4
Int'l Telephone & Telegraph	1.0
Kellogg Co.	0.3
Coca Cola Co.	2.6
Kraftco Corp.	1.0
Minnesota Mining & Mfg. Co.	3.1
Merck & Co. Inc.	5.6
Natl. Cash Register	0.3
Nalco Chemical Co.	2.1
Procter & Gamble Co.	3.2
Reynolds, R.J. Inds. Inc.	1.1
Sears, Roebuck & Co.	7.1
Standard Oil Co. of Calif.	2.7
Sunbeam Corp.	0.5
Square D Co.	0.1
Shell Oil Co.	5.9
Timken Co.	2.1
TRW Inc.	1.9
Texaco Inc.	6.3
Union Oil of California	1.7
Exxon Corp.	0.9
TOTAL	100.0%

multi-index model thus achieves efficient diversification while demonstrating superior ex-post performance with a positive alpha of 2.0 percent. In sum, it seems reasonable to conclude that the multi-index model not only has superior ex-ante, but also superior ex-post properties.

Test of Stability of Input Relationships

The final part of this section discusses the stability of input relationships to the multi-index model. This assessment is important, since the ease of practical implementation of the model depends on stability of the inputs over time. More specifically, the greater their stability, the more extensively historic data may be used as inputs, thus avoiding recourse to developing inputs by more arbitrary or expensive approaches. Recall that inputs required for the multi-index model are estimates for the four indexes of growth, cyclical, stable, and oil stocks, as well as estimates of the values relating individual stocks in a group with their corresponding index.

The analysis began by assessing the stability over time of the correlation relationships among the four groupings of growth, cyclical, stable, and oil stocks. This stability can be assessed by means of the index procedure used in the previous section to measure the degree to which the four stock groupings showed characteristics of high within-group correlation and low between-group correlation. Previously applied to the 1961-69 period, this procedure involved forming indexes from the stocks within each of the four groups and then correlating individual stocks with each of the four indexes. We applied this procedure to the sample stock data for the 1970-74 performance period and compared the results.

Figure 8 shows the results of the index procedure for the 1970-74 ex-post period (Figure 4 in the previous section shows the results for the 1961-69 period). The two figures display summary matrices showing the average value of the correlation coefficients of the stocks within each group (growth, cyclical, stable, oil) with the corresponding index, as well as with the other three indexes. Comparing the 1970-74 results to those of the earlier 1961-69 period, one sees the same desirable pattern of high within-group correlation, and low across-group correlation. The average values of the correlation coefficients between stocks and the index to which assigned were all positive and highly significant. In contrast, stocks showed a low degree of correlation with indexes other than the one to which assigned. These results indicate a fairly high degree of stability over time in the relationship among groupings.

The next step in the test of the stability of stock group relationships was to measure the extent over time to which stock indexes maintain a stable relationship to the general market factor. The returns of each of the four indexes of growth, cyclical, stable, and oil stocks were regressed on the S&P 425 return for both the 1961-69 and 1970-74 periods. The betas from these regressions were assumed to be broadly representative of the relationship between the respective stock groupings and the general market factor. The betas for the two periods were compared to assess their stability over time.

FIGURE 8

Matrix of Residual Correlation Coefficients
Averages of Stocks and Four Stock Indexes
(1970 - 1974)

	GROWTH INDEX	CYCLICAL INDEX	STABLE INDEX	OIL INDEX
GROWTH AVERAGE	.33	-.04	-.01	-.41
CYCLICAL AVERAGE	-.07	.37	.05	-.03
STABLE AVERAGE	-.04	.08	.37	-.03
OIL AVERAGE	-.21	-.02	-.03	.69

Table 8 shows the results of this test of beta stability. Note that the corresponding betas from the two periods are very similar, with the betas for the stable grouping almost identical. The statistic in the third column tests the difference in the betas between the periods. If there is in fact no difference, the value of the test statistic will exceed 2.00 five percent of the time. Note that the values of the test statistic are well below that figure, indicating that the correlation relationship among groupings was quite stable between 1961-69 and 1970-74.

Table 8

Regression Coefficients of Group Indexes vs. S&P 425
(1961-1969) and (1970-1974)

Group Index	Beta (1961-1969)	Beta (1970-1974)	Beta Test Statistic (*)
Growth	1.24	1.27	-.260
Stable	.90	.90	.110
Cyclical	1.11	1.04	.745
Oil	.85	.95	-.743

*Note: Value of ± 2.00 would indicate statistically significant at .05 level.

In order to use the four-index model, one needs return estimates for the four indexes of growth, cyclical, stable, and oil stocks, as well as estimates of specific return (A_i) and responsiveness (B_i) measures relating individual stocks to their respective indexes. When the responsiveness parameters were tested for stability by comparing their values in the 1961-69 and 1970-74 periods, the large majority were not significantly different from period to period—suggesting that use of historic values as estimates of future values is probably warranted.

The behavior of the alpha values over time, however, makes it inadvisable to use historic values as projections of future experience. A statistical test showed that the large majority of values were not significantly different from zero, suggesting that any non-zero historical values for this parameter should be generally considered as random noise or highly unreliable for projection purposes. Perhaps the best assumption to make when merely using historic data to develop projected values, is to assume that this parameter is zero.

VI. Summary and Conclusion

A prime purpose of this study was to develop homogeneous stock groupings suitable for use as inputs in a multi-index model. Several statistical techniques were employed in testing whether the residuals obtained by removing general market effects from a sample of 100 stocks displayed the pattern of cross-sectional dependence that would conform to the hypothesized three stock categories. These statistical procedures showed that the sample of 100 stocks could be classified into four, rather than three, distinct groups, with an oil group being the unexpected fourth group. In addition, regression analysis results indicated that, over the full period of the study, these stock groupings accounted for an average of 15 percent of the variance in rate of return of stocks in the sample, over and above the 30 percent explained by the general market factor. This result suggested that, in addition to market and industry factors, a set of explanatory factors based on classification of stocks according to: (1) growth, (2) stable, (3) cyclical, and (4) oil would be useful in explaining the variation in stock returns over time.

The existence of such stock groupings implies that residuals for the single-index model are not uncorrelated. An examination of the residual correlation matrix for this model confirmed the existence of a significant degree of dependence among the residuals, whereas an examination of this matrix for the four-index model showed very little dependence. The correlation matrix of stock returns implied by the four-index model provides a closer approximation to the true correlation matrix than is provided by the single-index model, hence accounting more satisfactorily for systematic effects among securities.

As a result, it was expected that portfolio selection based on the multi-index model would provide performance superior to selection based on the single-index model. Comparison of respective sets of efficient portfolios generated by the two models showed that the multi-index model provided portfolios with lower risk at equivalent return over a broad range of the efficient frontier. The multi-index model diversified more efficiently than the single-index model owing to the facility for distributing the portfolio more evenly across the four homogeneous groups of growth, cyclical, stable, and oil stocks.

The relative performance of the two models was subjected to ex-post test over the period 1970-74 using historical (1961-69) data as inputs to the models. The performance of these models was also compared to that of large all-stock mutual funds and the S&P 500. Both models outperformed the mutual funds, with the single-index model showing performance approximately in line with the S&P 500 over the 1970-74 period. The multi-index model, by outperforming both the single-index model and the S&P 500, displayed not only superior ex-ante, but also superior ex-post properties.

Finally, the study analyzed the stability of the parameters of the multi-index model. Tests indicated that, between the 1961-69 and 1970-74 periods, the growth, cyclical, stable, and oil groups maintained stable values for the relationships between individual stocks and their corresponding indexes. This stability suggests the possibility of extensive use of historic values as inputs, thereby avoiding recourse to more expensive or arbitrary approaches.

In conclusion, analysis of the characteristics of the multi-index portfolio selection model indicates that it has significant potential advantage over other types of portfolio selection models. It also appears to have a potential advantage over traditional practical portfolio management practice, as evidenced by the superiority of its performance to that of mutual funds. Finally, the stability of some of the input relationships indicates that the process of practical implementation may be easier than might be expected. The multi-index model should in fact be the preferred model for practical portfolio analysis.